Simultaneity, Radar 4-Coordinates and the 3+1 Point of View about Accelerated Observers in Special Relativity.

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Abstract

After a review of the 1+3 point of view on non-inertial observers and of the problems of rotating reference frames, we underline that was is lacking in their treatment is a good global notion of simultaneity due to the restricted validity (coordinate singularities show up) of the existing 4-coordinates associated to an accelerated observer (like the Fermi normal ones).

We show that the relativistic Hamiltonian 3+1 point of view, based on a 3+1 splitting of Minkowski space-time with a foliation whose space-like leaves are both simultaneity and Cauchy surfaces, allows to find a solution to such problems, if we take into account Møller's definition of allowed 4-coordinate transformations extended to radar 4-coordinates. Each admissible choice of simultaneity implies an associated definition of instantaneous 3-space (and of spatial distance) and of one-way velocity of light.

Rigidly rotating relativistic reference frames are shown not to exist. We give explicit foliations, with simultaneity surfaces (also space-like hyper-planes) non orthogonal to the arbitrary non-inertial observer world-line, which correspond to a good notion of simultaneity for suitable (mutually balancing) translational and rotational accelerations. Viceversa, given one such admissible foliation, we can determine the modification of Einstein's convention implied by its associated notion of simultaneity. This treatment allows:

i) To give the 3+1 description of both the rotating disk (its 3-geometry depends on the choice of simultaneity)) and the Sagnac effect.

ii) To show how a GPS system of spacecrafts may establish a grid of admissible radar 4-coordinates, namely an empirical notion of simultaneity.

iii) How, given an admissible empirical notion of simultaneity adapted to Earth's rotation, instead of assuming Einstein's convention plus Sagnac corrections, it is possible to determine the associated time delay (including the Shapiro delay as a post-Newtonian effect) between an Earth station and a satellite. Its comparison with the future measurements of the ACES mission will allow to synchronize the clocks according to this empirical simultaneity.

We show that in parametrized Minkowski theories all the admissible notions of simultaneity are gauge equivalent (*conventionality of simultaneity as a gauge theory*) and, as an example, we describe Maxwell theory in non-inertial systems with any admissible notion of simultaneity, like those needed for a correct treatment of the magnetosphere of pulsars. These considerations can be extended to canonical metric gravity on globally hyperbolic space-times, where, however, the admissible notions of simultaneity are *dynamically* determined by the ADM Hamilton equations, equivalent to Einstein;s equations.

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I. INTRODUCTION

The increasing importance of special relativity and post-Newtonian gravity in fields connected with space navigation and experiments in the Solar System, clock synchronization and, more in general, with the rotational aspects of relativistic kinematics in astrophysics requires a revisitation of the topics connected with the notion of simultaneity and with the problem of how an accelerated observer can build a good system of radar 4-coordinates compatible with a given notion of simultaneity. This paper is devoted to such a revisititation and to an attempt to find a unified treatment of these problems. Therefore we start with a review of the open problems and then we state our viewpoint.

We shall use the signature $\eta^{\mu\nu} = \boldsymbol{\epsilon}\,(+, - - -)$, with $\boldsymbol{\epsilon} = \pm$ according to whether the particle physics or general relativity convention is adopted, for the Minkowski 4-metric and we shall put $c = 1$. Nevertheless, we keep c in various formulas for the sake of clarity.

Newtonian mechanics in Galilei space-time (fusion of an absolute space with an absolute time) and special relativity in Minkowski space-time (an absolute space-time) rely both on a *relativity principle*, according to which the laws of physics are the same in every *inertial system* (an inertial observer with its time axis and a choice of space axes), namely in a special family of *rigid systems of reference* in uniform translational motion one with respect to the other. That is the laws of nature are covariant and there is no preferred inertial observer. In any inertial system, according to the law of inertia, a material particle, not acted upon by any agent, will continue to move in a straight line with constant velocity. Special coordinates are associated to each inertial system: either Cartesian 3-coordinates plus time or pseudo-Cartesian (Lorentzian) 4-coordinates. The transition from an inertial system to another one is performed with a kinematical group of global transformations, either the Galilei group or the Poincare' group.

Then, the empirical point of view needed to establish a theory of measurements requires the replacement of abstract ideal concepts like (either absolute or dynamical) *time* and *space* with actual metrological standards like a *clock* and a *rod*: physical time intervals and spatial distances are only relative quantities with respect to the chosen reference standard units (only ratios of quantities are physically meaningful), which are constantly upgraded following the developments of both theory and technology.

In Newtonian mechanics the absolute space may be identified as an inertial system asso-

ciated to the fixed stars. In special relativity, where it is the space-time to be absolute, one makes a conventional choice of a *quasi-inertial (non-rotating)* reference system [1, 2] [1]. In both theories the chrono-geometrical structure is absolute, i.e. non dynamical. Given some standard of length and time (*rods and clocks*), we can perform any measurement we like on a system independently from its dynamics. Only in general relativity the chrono-geometrical structure becomes dynamical [8, 9].

The main difference between the two theories lies in the notion of *simultaneity of two events*. Due to the absolute nature of Newtonian time, the points on a $t = const.$ section of Galilei space-time are all simultaneous (instantaneous absolute 3-space), whichever inertial system we are using. As a consequence, the causal notions of *before* and *after* a certain event are absolute. Instead in special relativity there is no absolute notion of simultaneity. Given an event, all the points outside the light cone with vertex in that event are not causally connected with that event (they have space-like separation from it), so that the notions of *before* and *after* an event become *observer-dependent*. Once we have chosen to describe physics with respect to any inertial system (x^μ are the associated Cartesian 4-coordinates), the events simultaneous for the inertial observer chosen as origin are usually assumed to be those lying in the space-like hyper-planes $x^o = c\,t = const.$ in accord with Einstein's convention for the synchronization of distant clocks.

As a consequence, the synchronization of two space-like separated clocks has to be defined,

[1] According to the IAU 2000 Resolutions [3], it is a coordinate system named the *Solar System Barycentric Celestial Reference System*, which is materialized in the *Solar System Barycentric Celestial Reference Frame* by specifying its axes by means of fixed stars (quasars) in the Hypparcos catalog [4]. For processes in the vicinity of the Earth the (non-inertial but non-rotating) *Geocentric Celestial Reference System* is used. In the definition of these coordinate systems the post-Newtonian approximation to general relativity is taken into account [5]. As a consequence the metrology of general relativity [6] (see Ref.[7] for an older point of view) has to be used to define measurable quantities (compatible with general covariance) inside the solar system, since the special relativistic approximation is no more sufficient to describe phenomena like time delays. In particular, given the time-like world-line of an observer, the *proper time* of the clock and the *proper length* of an infinitesimal rod, together with associated coordinate-independent units, carried by the observer, have to be defined. Assuming the value of the two-ways invariant velocity of light c as a conventional constant, the unit of length, the *proper meter*, is derived from the unit of time, the *proper second*. For the synchronization of distant clocks *Einstein's convention is used*, while for the definition of the distance of events at finite space-like separation both the world-line of an observer associated to one of these events and a space-like path joining them is needed besides the synchronization convention. In practice, special coordinate systems, like the two previous ones, are introduced and coordinate-dependent units of time (the TAI second) and of length are used, with suitable instructions to connect them to the proper units.

not being implied by the chrono-geometrical structure of Minkowski space-time. Usually this is done by means of *Einstein's convention* [10] (see for instance Refs.[11, 12]) based on the choice of the rays of light [2] as preferred tools to measure time and length. In a given inertial system the clock A, associated to the time-like world-line γ_A, emits a light signal at its time x^o_{Ai}, corresponding to an event Q_i on γ_A, towards the time-like world-line γ_B carrying the clock B [3]. When the signal arrives at a point P on γ_B, it is reflected towards γ_A, where it is detected at time x^o_{Af}, corresponding to an event Q_f on γ_A. Then the clock B at the event P on γ_B is synchronized to the time $x^o_A = \frac{1}{2}\,(x^o_{Ai} + x^o_{Af})$, corresponding to an event Q in between Q_i and Q_f. It can be checked that Q and P *lie on the same space-like hyper-plane orthogonal to the world-line* γ_A, i.e. that they are simultaneous events for the chosen inertial observer [4]. In general relativity Einstein's convention has been generalized to non-inertial observers by Martzke and Wheeler [24] to define local 4-coordinates, which can be called *radar coordinates* due to the technology implied to build them. An alternative to the use of light rays is the synchronization of clocks by their *slow transport*: we shall not deal with it,

[2] The *conformal* structure of Minkowski space-time is selected by the *two independent postulates* of special relativity that the round-trip velocity of light is the *same* in every inertial system (the round trip postulate) and *isotropic* (the light postulate). Let us remark that only the *round-trip* (*two-way*) speed of light has a physical significance, since the *one-way* velocity between two events A and B (and its being or not isotropic) depends on the definition of synchronization of the two clocks at those points, i.e. from the notion of simultaneity used.

[3] The clocks are assumed to be standard clocks measuring proper time. See Refs.[6, 13, 14] for their mathematical characterization in special and general relativity.

[4] Einstein's convention has been criticized by Reichenbach and Grundbaum [15], who said that any convention $x^o_A = E\, x^o_{Af} + (E-1)\, x^o_{Ai} = x^o_{Ai} + E\,(x^o_{Af} - x^o_{Ai})$ with $0 < E = const. < 1$ can be used (*conventionalism of simultaneity*) without leading to any contradiction. In general light propagation becomes anisotropic, i.e. direction dependent, because from $x^o_B = x^o_A$, $x^o_B - x^o_{Ai} = E\,(x^o_{Af} - x^o_{Ai})$, $x^o_{Af} - x^o_B = (1-E)\,(x^o_{Af} - x^o_{Ai})$ we get $c_{AiB} = \frac{c}{2}\,E$, $c_{BAf} = \frac{c}{2}\,(1-E)$, $\frac{2}{c} = \frac{1}{c_{AiB}} + \frac{1}{c_{BAf}}$ with either $c_{AiB} > c$ or $c_{BAf} > c$ (but not both). See Refs.[16, 17, 18, 19, 20, 21, 22] and their rich bibliography for the various aspects of the debate about the conventionalist point of view. In particular let us stress the following points: a) The constant E may be generalized to a point-dependent function (see for instance Ref.[23]); b) Anderson and Stedman [18] underline how the notions of spatial distance depend on the choice of the notion of simultaneity made by the observer (see footnote 1); c) Giulini [17] shows that in the relativistic case Malament's non-conventionalist notion of *absolute simultaneity* [16] (as an equivalence relation implied by the causal automorphisms preserving the light-cone structure) has to be replaced with a notion of *relative simultaneity* with respect to some additional structure on space-time; d) both Anderson-Stedman [18] and Minguzzi [22] propose a *gauge interpretation* of simultaneity and of the one-way velocity of light; e) in Ref.[19], where there is a review of the various viewpoints on the conventionality of simultaneity, it is underlined the conventional nature of the statements about the isotropy or anisotropy of light propagation and of the measurements of length.

since in Ref. [25] it is shown its equivalence to Einstein's convention notwithstanding claims of the contrary (see also Section 2.1 of Ref.[19]).

In both theories the concept of inertial observers is an *idealization*. Every actual observer is always accelerated and in practice (for instance in astronomy) one speaks of *quasi-inertial systems* [1, 2, 3, 4]. In Newtonian mechanics they are defined as those *rigid* systems of reference, in which the sensibility of the measuring apparatuses does not allow to detect any inertial force, like the Coriolis one, which modify Newton's law ($m\,\vec{a} = \vec{F} \;\mapsto\; m\,\vec{a}' = \vec{F}' + m\,\vec{f}$) when dynamics is described by an observer carrying a rigid accelerated system of reference (see for instance Ref.[26], Section 39). In special relativity the notion of quasi-inertial system is more problematic, because there is no accepted definition of inertial forces seen by an accelerated observer when a manifestly Lorentz covariant description of relativistic mechanics is used [5]. It is only in the Hamiltonian version of relativistic mechanics that we can re-introduce a notion of non-inertial forces, because Hamilton equations define a force law.

Due to the absence of any statement about non-inertial systems (replacing the relativity principle), usually special relativity is seen as an approximation to general relativity, valid locally near an observer in free fall [6]. The equivalence principle is invoked to say that an uniform gravitational field and an uniform acceleration are locally indistinguishable [7] and that it is meaningless to speak of inertial forces in general relativity: but again at the Hamiltonian level this is possible with respect to non-rigid systems of reference as shown in Ref.[8]. Since the transition to general relativity adds new problems without solving the special relativistic ones, let us concentrate our discussion on special relativity without gravity as an autonomous theory.

[5] See Ref.[27] for the problems, like the no-interaction-theorem, associated to the description of the motion of massive relativistic particles either free or with action-at-a-distance relativistic interactions. Again the main problem is how to perform a simultaneous description of the interacting particles.

[6] Also general relativity makes no positive statement about non-inertial systems: the laws of nature are now *generally covariant* (namely they assume the same form in every 4-coordinate system), but this *only* implies the elimination of rigid inertial systems (only local inertial systems for an observer in free fall remain). Moreover, now the chrono-geometrical structure of space-time becomes *dynamical* (it is described by the metric tensor, which is also the potential for the gravitational field), space-time itself looses its reality and we need a physical identification of space-time points as point events (*space-time is the gravitational field itself*). See Ref.[8, 9] for a full discussion of these aspects of general relativity.

[7] But more realistically (see Ref.[28]) this is true only on the geodesic of an observer in free fall, due to the gravitational tidal forces evidentiated by the geodesic deviation equation.

A. The Locality Hypothesis.

Since the actual observers are accelerated, we need some statement correlating the measurements made by them to those made by inertial observers, the only ones with a general framework for the interpretation of their experiments. This statement is usually the *hypothesis of locality* which is expressed in the following terms according to Mashhoon [29] (see also Refs.[30, 31]): *an accelerated observer at each instant along its world-line is physically equivalent to an otherwise identical momentarily comoving inertial observer*, namely a non-inertial observer passes through a continuous infinity of hypothetical momentarily comoving inertial observers [8]. While this hypothesis is verified in Newtonian mechanics and in those relativistic cases in which a phenomenon can be reduced to point-like coincidences of classical point particles and light rays (geometrical optic approximation), its validity is questionable in presence of electro-magnetic waves. As emphasized by Mashhoon [29, 30, 31], in this case we can trust the locality hypothesis only when the wave-length λ of the electro-magnetic wave is much shorter of the acceleration length $\mathcal{L}$ of the observer, describing the degree of variation of its state [9], i.e. when $\lambda << \mathcal{L}$. When $\lambda << \mathcal{L}$ holds, so that the period of the wave satisfies $\frac{\lambda}{c} << \frac{\mathcal{L}}{c}$, the observer state does not change appreciably on the time scale needed to detect a few oscillations of the wave and to measure its frequency. Instead in the case of the electro-magnetic waves radiated by an accelerating charged particle with acceleration length $\mathcal{L}$, we have $\lambda \approx \mathcal{L}$. In this case it is highly problematic to consider the particle momentarily equivalent to an identical comoving inertial particle. This fact is confirmed by the causality problems (pre-acceleration, runaway solutions) of the classical Abraham - Lorentz - Dirac equation of motion of the particle (see for instance Ref.[34]), which depend on the time derivative of the acceleration, and by the still going on discussions [35] on the energy balance and the back-reaction in these radiative phenomena, due, besides the problem of the self-energies, to the absence of a clear notion of simultaneity for the particle and electro-magnetic degrees of freedom allowing to define a well posed Cauchy problem [10].

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[8] For Einstein's comments on this point see Stachel [32]. Møller ([12], p.223) makes the assumption that the length of the measuring rods are independent of the accelerations relative to an inertial system. For Klauber [33] it is the *surrogate frame postulate*.

[9] $\mathcal{L} = \frac{c^2}{a}$ for an observer with translational acceleration a; $\mathcal{L} = \frac{c}{\Omega}$ for an observer rotating with frequency Ω.

[10] See Refs. [36] for a semi-classical Hamiltonian approach to these problems by using Grassmann-valued electric charges to regularize the self-energies.

Also the measurement of time dilation based on the muon lifetime can be shown [29] to give the standard result $\tau_{(\mu)} = \gamma\, \tau^o_{(\mu)}$ ($\tau^o_{(\mu)}$ is the lifetime in an inertial system) only modulo negligible corrections of order $(\lambda/\mathcal{L})^2$ ($\lambda = \hbar/m\, c$ is the muon Compton wavelength). The hypothesis of locality is clearly valid in many Earth-based experiments since $c^2/g_{Earth} \approx 1\, lyr$, $c/\Omega_{Earth} \approx 20\, AU$.

As we shall see, there are simultaneity conventions which satisfy the locality hypothesis and others in which the associated observers are not a sequence of comoving inertial ones. Only in a theory in which all the simultaneity conventions are equivalent in some sense the locality hypothesis can be fully justified.

B. The 1+3 Point of View.

Therefore it is far from clear which is the description of physical phenomena given by a non-inertial observer, especially a rotating one. The fact that we can describe phenomena only locally near the observer and that the actual observers are accelerated leads to the *1+3 point of view* (or *threading splitting*) [37, 38].

Given the world-line γ of the accelerated observer, we describe it with Lorentzian coordinates $x^\mu(\tau)$, parametrized with an affine parameter τ, with respect to a given inertial system. Its unit 4-velocity is $u^\mu_\gamma(\tau) = \dot x^\mu(\tau)/\sqrt{\epsilon\, \dot x^2(\tau)}$ [$\dot x^\mu = \frac{dx^\mu}{d\tau}$]. The observer proper time $\tau_\gamma(\tau)$ is defined by $\epsilon\, \dot{\tilde x}^2(\tau_\gamma) = 1$, if we use the notations $x^\mu(\tau) = \tilde x^\mu(\tau_\gamma(\tau))$ and $u^\mu(\tau) = \tilde u^\mu(\tau_\gamma(\tau)) = d\, \tilde x^\mu(\tau_\gamma)/d\, \tau_\gamma$, and it is indicated by a *comoving standard atomic clock*. At each point of γ with proper time $\tau_\gamma(\tau)$, the tangent space to Minkowski space-time in that point has the 1+3 splitting in vectors parallel to $\tilde u^\mu(\tau_\gamma)$ and vectors lying in the 3-dimensional (so-called *local observer rest frame*) subspace $R_{\tilde u(\tau_\gamma)}$ orthogonal to $\tilde u^\mu(\tau_\gamma)$ [11]. By a conventional choice of three spatial axes $E^\mu_{(a)}(\tau) = \tilde E^\mu_{(a)}(\tau_\gamma(\tau))$, $a = 1, 2, 3$, orthogonal to $u^\mu(\tau) = E^\mu_{(o)}(\tau) = \tilde u^\mu(\tau_\gamma(\tau)) = \tilde E^\mu_{(o)}(\tau_\gamma(\tau))$, the non-inertial observer is endowed with an ortho-normal tetrad [12] $E^\mu_{(\alpha)}(\tau) = \tilde E^\mu_{(\alpha)}(\tau_\gamma(\tau))$, $\alpha = 0, 1, 2, 3$.

[11] Let us remark that there is *no notion of a 3-space simultaneous with a point of* γ and whose tangent space at that point is $R_{\tilde u(\tau_\gamma)}$: i) at a geometrical level $R_{\tilde u(\tau_\gamma)}$ may at best be considered as a local approximate substitute of it in an infinitesimal neighborhood; ii) actually, when the locality hypothesis holds, the acceleration radii determine the effective dimension of the neighborhood and this fact has led some authors, see for instance Ref.[21], to identify $R_{\tilde u(\tau_\gamma)}$ with the simultaneity 3-space of the observer.

[12] This amounts to a choice of three comoving gyroscopes in addition to the comoving standard atomic clock.

The matter or field tensors seen by the observer are the (coordinate-independent) tetradic components of these tensors: for the electro-magnetic field strength $F_{\mu\nu}(z)|_{z=\tilde{x}(\tau_\gamma)}$ they are $F_{(\alpha)(\beta)}(\tilde{x}(\tau_\gamma)) = F_{\mu\nu}(\tilde{x}(\tau_\gamma))\, \tilde{E}^\mu_{(\alpha)}(\tau_\gamma)\, \tilde{E}^\nu_{(\beta)}(\tau_\gamma)$. In the case of a vector field $X^\mu(z)$, the physical observables [33] for the observer are the scalar quantities formed with $X_{(o)}(\tau_\gamma) = X_\mu(\tilde{x}(\tau_\gamma))\, \tilde{E}^\mu_{(o)}(\tau_\gamma) = X_\mu(\tilde{x}(\tau_\gamma))\, \tilde{u}^\mu(\tau_\gamma)$ and $\sum_{(a)} X^2_{(a)}(\tau_\gamma)$ with $X_{(a)}(\tau_\gamma) = X_\mu(\tilde{x}(\tau_\gamma))\, \tilde{E}^\mu_{(a)}(\tau_\gamma)$. For instance, if we have two incident light rays with tangent vectors k^μ, $k^{'\mu}$ [$\eta_{\mu\nu}\, k^\mu\, k^\nu = \eta_{\mu\nu}\, k^{'\mu}\, k^{'\nu} = 0$], their observable angle ϕ_γ seen by the observer is defined by $cos\, \phi_\gamma = \sum_{(a)} k_{(a)}\, k^{'}_{(a)}/|k_{(o)}|\, |k^{'}_{(o)}|$.

However the threading point of view says nothing on how to define sets of events of Minkowski space-time *simultaneous* with each point of the world-line γ of the non-inertial observer and this is a source of problems both for the synchronization of clocks and for the definition of measurements of length. Let us see what is known in the literature.

C. 4-Coordinates for Accelerated Observers.

A) A first attempt (both in special and general relativity) to treat this problem is based on the *Fermi normal coordinates* [40]. In each point of the world-line γ of the accelerated observer one considers the *hyper-plane orthogonal* to the observer unit 4-velocity $u^\mu(\tau)$, i.e. the local observer rest frame at that point. Then, on each hyper-plane one considers three space-like geodesics as spatial coordinate lines. In this way we can coordinatize a world-tube around the world-line γ, whose *radius* is determined by the intersection of hyper-planes at different times. Notwithstanding various efforts to ameliorate the construction [41], in this way we obtain only local coordinates and a notion of simultaneity valid only inside the world-tube (see also Section 4.1 of Ref.[19]). Let us remark that similar coordinates are employed in the attempts to define the relativistic center of mass of an extended object [42] [13].

B) Mashhoon [29, 31] has introduced a variant of the previous coordinates, i.e. the

Usually the spatial axes are chosen to be Fermi-Walker transported as a standard of non-rotation, which takes into account the Thomas precession [39]. Let us remark that this physical reference frame of an observer has not to be confused with the Celestial Reference Frames (see footnote 1) used in astronomy and geodesy.

[13] See Refs.[43] for the study of the problem of the relativistic center of mass without using this type of coordinates.

geodesic coordinates for rotating observers. Since in this construction the observer has spatial axes $E^\mu_{(a)}(\tau)$ obtained from those of an inertial observer with a Lorentz boost to a comoving observer along the axis 2 (tangent to the circular orbit), the observer can define an *acceleration tensor* $\mathcal{A}_{(\alpha)(\beta)}$ by means of the equation $\frac{d\, E^\mu_{(\alpha)}(\tau)}{d\, \tau} = \mathcal{A}_{(\alpha)}{}^{(\beta)}(\tau)\, E^\mu_{(\beta)}(\tau)$. The translational acceleration and rotational frequency of the observer are $a_{(i)} = \mathcal{A}_{(o)(i)}$ and $\Omega_{(i)} = \frac{1}{2}\, \epsilon_{(i)(j)(k)}\, \mathcal{A}_{(j)(k)}$, respectively. The two invariants $I_1 = \vec{\Omega}^2 - \vec{a}^2$ and $I_2 = \vec{a} \cdot \vec{\Omega}$ may be interpreted as the two *acceleration radii* determining the world-tube of validity of these geodesic coordinates, namely the region where the hyper-planes orthogonal to the observer world-line can be considered as simultaneity 3-spaces (see footnote 11).

C) A third approach is the one of Pauri and Vallisneri [44] [14] . It is a refinement of the Martzke-Wheeler method [24] to build radar coordinates. Given the observer world-line $x^\mu(\tau)$, the simultaneity surface through $x^\mu(0)$ is built as the union of the intersections of the past light-cone of $x^\mu(+\Delta\tau)$ with the future light-cone of $x^\mu(-\Delta\tau)$ when $\Delta\tau$ varies (if the world-line is a straight line one recovers the hyper-planes of Einstein's convention). In this way it is possible to build a foliation of Minkowski space-time with space-like hyper-surfaces (simultaneity surfaces). However, in the example explicitly worked out by these authors the embedding $x^\mu = z^\mu(\tau, |\vec{\sigma}|\, \hat{n})$ ($|\vec{\sigma}|$ is the radial distance from γ and $\hat{n}$ a unit 3-vector) describing these hyper-surfaces in Minkowski space-time is a periodic function of $|\vec{\sigma}|$ with an oscillating limit for $|\vec{\sigma}| \to \infty$. Again this signals the presence of a coordinate singularity after one spatial period and a limited range of validity of these coordinates too.

D. The Rotating Disk.

Finally there is the enormous amount of bibliography, reviewed in Ref.[45], about the problems of the *rotating disk* and of the rotating coordinate systems. Independently from the fact whether the disk is a material extended object or a geometrical congruence of time-like world-lines (integral lines of some time-like unit vector field), the idea followed by many researchers [11, 12, 46] is to start from the Cartesian 4-coordinates of a given inertial system, to pass to cylindrical 3-coordinates and then to make a either Galilean (assuming a non-relativistic behaviour of rotations at the relativistic level) or Lorentz transformation to

[14] See this paper, Ref.[20] and their bibliography for the role of the notion of simultaneity in the interpretation of the twin paradox. In particular in Ref.[20] it is shown the independence of the solution of the paradox from the choice of the notion of simultaneity.

comoving rotating 4-coordinates, with a subsequent evaluation of the 4-metric in the new coordinates. In other cases [47] a suitable global 4-coordinate transformation is postulated, which avoids the so-called *horizon problem* (the points where all the previous 4-metrics have either vanishing or diverging components, when the rotational frequency reaches the velocity of light). Various authors (see for instance Refs.[48]) do not define a coordinate transformation but only a rotating 4-metric. Just starting from Møller rotating 4-metric [12], Nelson [49] was able to deduce a 4-coordinate transformation implying it.

The problems arise when one tries to define measurements of length, in particular that of the circumference of the disk. Einstein [50] claims that while the rods along the radius R_o are unchanged those along the rim of the disk are Lorentz contracted: as a consequence more of them are needed to measure the circumference, which turns out to be greater than $2\pi\, R_o$ (non-Euclidean 3-geometry even if Minkowski space-time is 4-flat) and not smaller. This was his reply to Ehrenfest [51], who had pointed an inconsistency in the accepted special relativistic description of the disk [15] in which it is the circumference to be Lorentz contracted: as a consequence this fact was named the *Eherenfest paradox* (see the historical paper of Grøn in Ref.[45]).

As underlined in Ref. [52] (see also Refs.[53, 54]) there is an intertwining of the following problems: i) Does the rim of the disk in the rotating system define the same geometrical circumference as the set of the positions of the points of the rim as seen by a given inertial observer? ii) How is defined the *instantaneous 3-space of the rotating disk* if we take into account that the associated congruence of time-like world-lines describing its points in time has not zero vorticity, so that it is not surface-forming? iii) How to define the 3-geometry of the rotating disk? iv) How to measure the length of the circumference? v) Which time and notion of simultaneity has to be used to evaluate the velocity of (massive or massless) particles in uniform motion along the circumference? vi) Do standard rods undergo Lorentz contraction (validity of the locality hypothesis)? In Refs. [45, 55] there is a rich bibliography on the existing answers to these questions and the remark that the actual standard rods used for measurements are rods with free ends (not be confused with arcs of circumference), which a) remain unchanged when slowly transported; b) are assumed not to be influenced

[15] If R and R_o denote the radius of the disk in the rotating and inertial frame respectively, then we have $R = R_o$ because the velocity is orthogonal to the radius. But the circumference of the rim of the disk is Lorentz contracted so that $2\pi\, R < 2\pi\, R_o$ inconsistently with Euclidean geometry.

by an acceleration (the locality hypothesis) or a local gravitational field. In Section VIB more details on the rotating disk are given.

E. The Sagnac Effect.

The other important phenomenon connected with the rotating disk is the *Sagnac effect* [56] (see the recent review in Ref.[57] for how many interpretations of it exist), namely the phase difference generated by the difference in the time needed for a round-trip by two light rays, emitted in the same point, one co-rotating and the other counter-rotating with the disk [16]. This effect, which has been tested (see the bibliography of Refs.[57, 61]) for light, X rays and matter waves (Cooper pairs, neutrons, electrons and atoms), has important technological applications and must be taken into account for the relativistic corrections to space navigation, has again an enormous number of theoretical interpretations (both in special and general relativity) like for the solutions of the Ehrenfest paradox. Here the lack of a good notion of simultaneity leads to problems of *time discontinuities or desynchronization effects* when comparing clocks on the rim of the rotating disk.

Moreover, various authors use the Sagnac effect, together with the Foucault pendulum, as a clear hint that, contrary to translations, the rotations of the reference frame have an *absolute character* so that non-rotating frames are preferred frames [33] [17]. Another disturbing aspect of rotating frames for these authors is that the (coordinate) velocity of light is no more isotropic (see footnote 4) when the rotating 4-metric is *not time-orthogonal*, i.e. when $g_{oi} \neq 0$. In general relativity, where the g_{oi}'s are the (gauge, i.e. not determined

[16] For monochromatic light in vacuum with wavelength λ the fringe shift is $\delta z = 4\,\vec{\Omega}\cdot\vec{A}/\lambda\,c$, where $\vec{\Omega}$ is the Galilean velocity of the rotating disk supporting the interferometer and $\vec{A}$ is the vector associated to the area $|\vec{A}|$ enclosed by the light path. The time difference is $\delta t = \lambda\,\delta z/c = 4\,\vec{\Omega}\cdot\vec{A}/c^2$, which agrees, at the lowest order, with the proper time difference $\delta\tau = (4\,A\,\Omega/c^2)\,(1-\Omega^2\,R^2/c^2)^{-1/2}$, $A = \pi\,R^2$, evaluated in an inertial system with the standard rotating disk coordinates. This proper time difference is twice the time lag due to the *synchronization gap* predicted for a clock on the rim of the rotating disk with a non-time orthogonal metric. See Refs.[57, 58, 59] for more details. See also Ref.[60] for the corrections included in the GPS protocol to allow the possibility of making the synchronization of the entire system of ground-based and orbiting atomic clocks in a reference local inertial system. Since usually, also in GPS, the rotating coordinate system has $t^{'} = t$ (t is the time of an inertial observer on the axis of the disk) the gap is a consequence of the impossibility to extend Einstein's convention of the inertial system also to the non-inertial one rotating with the disk: after one period two nearby synchronized clocks on the rim are out of synchrony.

[17] Let us remark that this is an attitude opposite to that of the supporters of Mach principle [62].

by Einstein's equations) *shift functions*, this fact implies the addition of *gravito-magnetic effects* (dragging of inertial frames, Lense-Thirring effect) [63, 64] to the anisotropy of light propagation (not to mention new phenomena like the gravitomagnetic clock effects [65, 66] and the spin-rotation couplings [67]; see the review [68]). Let us remark that in general relativity synchronous 4-coordinates, for which the shift functions vanish, are subject to develop singularities in short times, when one attempts to do numerical gravity.

Another area which is in a not well established form is electrodynamics in non-inertial systems either in vacuum or in material media (*problem of the non-inertial constitutive equations*). Its clarification is needed both to derive the Sagnac effect from Maxwell equations without gauge ambiguities [58] and to determine which types of experiments can be explained by using the locality principle to evaluate the electro-magnetic fields in the comoving system (see the Wilson experiment [69] and the associated controversy [70] on the validity of the locality principle) without the need of a more elaborate treatment like for the radiation of accelerated charges. It would also help in the tests of the validity of special relativity (for instance on the possible existence of a preferred frame) based on Michelson-Morley - type experiments [71] (see also Ref.[68]).

We do not accept the interpretation of rotations as absolute and refuse the points of view implying deviations from standard special relativity like the new postulates (no Lorentz contraction under rotations and preferred nature of non-rotating frames) of Klauber [33] or of Selleri [72]. Instead (see also Ref.[58]) we remark that the Sagnac effect and the Foucault pendulum are *experiments which signal the rotational non-inertiality of the frame*. The same is true for neutron interferometry [73], where different settings of the apparatus are used to *detect either rotational or translational non-inertiality of the laboratory*. As a consequence a null result of these experiments can be used to give a definition of *relativistic quasi-inertial system*.

Let us remark that the disturbing aspects of rotations are rooted in the fact that there is a deep difference between translations and rotations at every level both in Newtonian mechanics and special relativity: the generators of translations satisfy an Abelian algebra, while the rotational ones a non-Abelian algebra. As shown in Refs.[42, 74], at the Hamiltonian level we have that the translation generators are the three components of the momentum, while the generators of rotations are a pair of canonical variables (L^3 and $arctg\,\frac{L^2}{L^1}$) and an unpaired variable ($|\vec{L}|$). As a consequence we can separate globally the motion of the

3-center of mass of an isolated system from the relative variables, but we cannot separate in a global and unique way three Euler angles describing an overall rotation, because the residual vibrational degrees of freedom are not uniquely defined.

Let us also remark that general relativity has been completely re-expressed in the 1+3 point of view (starting from the works of Cattaneo [75] and ending with Refs.[76]): the real open problem of this approach is that no one is able to formulate a Cauchy problem in this setting.

In conclusion the 1+3 point of view has to face a big group of problems most of which originated by the absence of a good notion of simultaneity. They are not academic theoretical problems. The Global Positioning System [60] (GPS and its European counterpart Galileo) and the future mission ACES of the European Space Agency on the synchonization of clocks [77], due to the level of time keeping accuracy of the order 10^{-15} (and higher) reached by the standard laser cooled atomic clocks [78], requires to take into account relativistic effects till the order $1/c^3$ [79, 80, 81]. But this has to be done after the introduction of a good notion of simultaneity in general relativity, which becomes a special relativistic notion of simultaneity in presence of weak gravitational fields compatible with the (non-inertial non-rotating) Geocentric Celestial Reference System and the (inertial non-rotating) Solar System Barycentric Celestial Reference System. This fact, together with the increasing interest in astronomy for relativistic corrections to reference frames and light propagation [82] and in astrophysics for relativistic rotating stars and black holes [83], points to the necessity of a re-formulation of the previous problems in a framework allowing a good control of the notion of simultaneity.

F. The 3+1 Point of View, Møller Admissible Coordinates and Parametrized Minkowski Theories.

The aim of this paper is to try to show that the framework, in which these problems find a natural co-existence with the standard treatment of special relativity, requires an inversion of attitude. Let us consider the 3+1 splittings of Minkowski space-time associated to its foliations with arbitrary space-like hyper-surfaces and not only with space-like hyper-planes. Each of these hyper-surfaces is both *a simultaneity surface and a Cauchy surface* for the equations of motion of the relativistic systems of interest. After the choice of a foliation, i.e. of a notion of simultaneity, we can determine, as we shall see, which are the

non-inertial observers compatible with that notion of simultaneity. Having given a *notion of simultaneity*, there will be associated notions of *one-way velocity of light*, of *synchronization of distant clocks*, of *instantaneous 3-space* and of *spatial length*.

Moreover in this framework we will show that it is possible

a) to give an operational method, generalizing Einstein's convention, for building the radar coordinates adapted to an arbitrary foliation [18] (see Subsection A of Section VI).

b) to solve the following inverse problem: given a single non-inertial observer or a congruence of non-inertial observers (like in the case of the rotating disk) find which are the foliations, i.e. the notions of simultaneity, compatible with them (see Sections III and V, respectively).

The 3+1 point of view is less physical (it is impossible to control the initial data on a non-compact space-like Cauchy surface), but it is the only known way to establish a well posed Cauchy problem for the dynamics, so to be able to use the mathematical theorems on the existence and uniqueness of the solutions of field equations for identifying the predictability of the theory. A posteriori, a non-inertial observer can try to separate the part of the dynamics, implied by these solutions, which is determined at each instant from the (assumed known) information coming from its causal past (see Ref.[85] for an attempt to re-phrase the instant form of dynamics in a form employing only data from the causal past of an observer) from the part coming from the rest of the universe.

As emphasized by Havas [23], to implement this program we have to come back to Møller's formalization [12] (Chapter VIII, Section 88) of the notion of simultaneity, based on previous work by Hilbert [86]. Given an inertial system with Cartesian 4-coordinates x^μ in Minkowski space-time and with the $x^o = const.$ simultaneity hyper-planes, Møller defines the *admissible coordinates transformations* $x^\mu \mapsto y^\mu = f^\mu(x)$ [with inverse transformation $y^\mu \mapsto x^\mu = h^\mu(y)$] as those transformations whose associated metric tensor $g_{\mu\nu}(y) = \frac{\partial h^\alpha(y)}{\partial y^\mu} \frac{\partial h^\beta(y)}{\partial y^\nu} \eta_{\alpha\beta}$ satisfies the following conditions

[18] See Refs.[84] for epistemological and mathematical supports to the notion of *radar coordinates*.

$$
\begin{aligned}
&\boldsymbol{\epsilon}\, g_{oo}(y) > 0, \\
&\boldsymbol{\epsilon}\, g_{ii}(y) < 0, \qquad \begin{vmatrix} g_{ii}(y) & g_{ij}(y) \\ g_{ji}(y) & g_{jj}(y) \end{vmatrix} > 0, \qquad \boldsymbol{\epsilon}\, det\,[g_{ij}(y)]\, < 0, \\
&\Rightarrow det\,[g_{\mu\nu}(y)]\, < 0. \qquad (1.1)
\end{aligned}
$$

These are the necessary and sufficient conditions for having $\frac{\partial h^\mu(y)}{\partial y^o}$ behaving as the velocity field of a relativistic fluid, whose integral curves, the fluid flux lines, are the world-lines of time-like observers. Eqs.(1.1) say:

i) the observers are time-like because $\boldsymbol{\epsilon} g_{oo} > 0$;

ii) that the hyper-surfaces $y^o = f^o(x) = const.$ are good space-like simultaneity surfaces.

Moreover we must ask that $g_{\mu\nu}(y)$ tends to a finite limit at spatial infinity on each of the hyper-surfaces $y^o = f^o(x) = const.$ If, like in the ADM canonical formulation of metric gravity [87], we write $g_{oo} = \boldsymbol{\epsilon}\,(N^2 - g_{ij}\, N^i\, N^j)$, $g_{oi} = g_{ij}\, N^j$ introducing the lapse (N) and shift (N^i) functions, this requirement says that the lapse function (i.e. the proper time interval between two nearby simultaneity surfaces) and the shift functions (i.e. the information about which points on two nearby simultaneity surfaces are connected by the so-called *evolution vector field* $\frac{\partial h^\mu(y)}{\partial y^o}$) do not diverge at spatial infinity. This implies that at spatial infinity on each simultaneity surface there is *no asymptotic either translational or rotational acceleration* [19] and the asymptotic line element is $ds^2 = g_{\mu\nu}(y)\, dy^\mu\, dy^\nu \rightarrow_{spatial\, infinity} \boldsymbol{\epsilon}\, \Big(F^2(y^o)\, (dy^o)^2 + 2\, G_i(y^o)\, dy^o\, dy^i - d\vec{y}^2\Big)$. But this would break manifest covariance unless $F(y^o) = 1$ and $G_i(y^o) = 0$. As a consequence, *the simultaneity surfaces must tend to space-like hyper-planes at spatial infinity.*

In this way all the admissible notions of simultaneity of special relativity are formalized as 3+1 splittings of Minkowski space-time by means of foliations whose leaves are space-like hyper-surfaces tending to hyper-planes at spatial infinity. Let us remark that admissible coordinate transformations $x^\mu \mapsto y^\mu = f^\mu(x)$ constitute the most general extension of the Poincare' transformations $x^\mu \mapsto y^\mu = a^\mu + \Lambda^\mu{}_\nu\, x^\nu$ compatible with special relativity. A

[19] They would contribute [88] to the asymptotic line element with the diverging terms $[A_i(y^o)\, y^i + B_{ij}(y^o)\, y^i\, y^j]\, (dy^o)^2$ and $\epsilon_{ijk}\, \omega^j(y^o)\, y^k\, dy^o\, dy^i$, respectively.

special family of admissible transformations are the sub-group of the *frame-preserving* ones: $x^o \mapsto y^o = f^o(x^o, \vec{x})$, $\vec{x} \mapsto \vec{y} = \vec{f}(\vec{x})$.

It is then convenient to describe [27, 88, 89] the simultaneity surfaces of an admissible foliation (3+1 splitting of Minkowski space-time) with *naturally adapted Lorentz scalar admissible coordinates* $x^\mu \mapsto \sigma^A = (\tau, \vec{\sigma}) = f^A(x)$ [with inverse $\sigma^A \mapsto x^\mu = z^\mu(\sigma) = z^\mu(\tau, \vec{\sigma})$] such that:

i) the scalar time coordinate τ labels the leaves Σ_τ of the foliation ($\Sigma_\tau \approx R^3$);

ii) the scalar curvilinear 3-coordinates $\vec{\sigma} = \{\sigma^r\}$ on each Σ_τ are defined with respect to an arbitrary time-like centroid $x^\mu(\tau)$ chosen as their origin;

iii) if $y^\mu = f^\mu(x)$ is any admissible coordinate transformation describing the same foliation, i.e. if the leaves Σ_τ are also described by $y^o = f^o(x) = const.$, then, modulo reparametrizations, we must have $y^\mu = f^\mu(z(\tau, \vec{\sigma})) = \tilde{f}^\mu(\tau, \vec{\sigma}) = A^\mu{}_A\, \sigma^A$ with $A^o{}_\tau = const.$, $A^o{}_r = 0$, so that we get $y^o = const.\, \tau$, $y^i = A^i{}_A(\tau, \vec{\sigma})\, \sigma^A$. Therefore, modulo reparametrizations, the τ and $\vec{\sigma}$ adapted admissible coordinates are *intrinsic coordinates*, which are mathematically allowed as charts in the atlas for Minkowski space-time. They are called *radar-like 4-coordinates* (see Subsection A of Section VI for the justification of this name) and, probably, they were introduced for the first time by Bondi [90]. The use of adapted Lorentz-scalar radar coordinates allows to avoid all the (often ambiguous) technicalities connected with the extrapolations of effects like Lorentz-contraction or time dilation from inertial to non-inertial frames.

The use of these Lorentz-scalar adapted coordinates allows to make statements depending only on the foliation but not on the 4-coordinates y^μ used for Minkowski space-time.

The simultaneity hyper-surfaces Σ_τ are described by their embedding $x^\mu = z^\mu(\tau, \vec{\sigma})$ in Minkowski space-time [$(\tau, \vec{\sigma}) \mapsto z^\mu(\tau, \vec{\sigma})$, $R^3 \mapsto \Sigma_\tau \subset M^4$] and the induced metric is $g_{AB}(\tau, \vec{\sigma}) = z^\mu_A(\tau, \vec{\sigma})\, z^\nu_B(\tau, \vec{\sigma})\, \eta_{\mu\nu}$ with $z^\mu_A = \partial z^\mu / \partial \sigma^A$ (they are flat tetrad fields over Minkowski space-time). Since the vector fields $z^\mu_r(\tau, \vec{\sigma})$ are tangent to the surfaces Σ_τ, the time-like vector field of normals $l^\mu(\tau, \vec{\sigma})$ is proportional to $\epsilon^\mu{}_{\alpha\beta\gamma}\, z^\alpha_1(\tau, \vec{\sigma})\, z^\beta_2(\tau, \vec{\sigma})\, z^\gamma_3(\tau, \vec{\sigma})$. Instead the time-like evolution vector field is $z^\mu_\tau(\tau, \vec{\sigma}) = N(\tau, \vec{\sigma})\, l^\mu(\tau, \vec{\sigma}) + N^r(\tau, \vec{\sigma})\, z^\mu_r(\tau, \vec{\sigma})$, so that we have $dz^\mu(\tau, \vec{\sigma}) = z^\mu_\tau(\tau, \vec{\sigma})\, d\tau + z^\mu_r(\tau, \vec{\sigma})\, d\sigma^r = N(\tau, \vec{\sigma})\, d\tau\, l^\mu(\tau, \vec{\sigma}) + (N^r(\tau, \vec{\sigma})\, d\tau + d\sigma^r)\, z^\mu_r(\tau, \vec{\sigma})$.

Since the 3-surfaces Σ_τ are *equal time* 3-spaces with all the clocks synchronized, the spatial distance between two equal-time events will be $dl_{12} = \int_1^2 dl\, \sqrt{{}^3g_{rs}(\tau, \vec{\sigma}(l))\, \frac{d\sigma^r(l)}{dl}\, \frac{d\sigma^s(l)}{dl}}$ [$\vec{\sigma}(l)$

is a parametrization of the 3-geodesic γ_{12} joining the two events on Σ_τ]. Moreover, by using test rays of light we can define the *one-way* velocity of light between events on different Σ_τ's.

When we have a Lagrangian description of an isolated system on arbitrary space-like hyper-surfaces (the *parametrized Minkowski theories* of Refs.[27, 89] and of the Appendix of Ref.[88]), *the physical results about the system do not depend on the choice of the notion of simultaneity*. In this approach, besides the configuration variables of the isolated system, there are the embeddings $z^\mu(\tau, \vec{\sigma})$ as extra *gauge configuration* variables in a suitable Lagrangian determined in the following way. Given the Lagrangian of the isolated system in the Cartesian 4-coordinates of an inertial system, one makes the coupling to an external gravitational field and then replaces the external 4-metric with $g_{AB} = z^\mu_A\, \eta_{\mu\nu}\, z^\nu_B$. Therefore the resulting Lagrangian depends on the embedding through the associated metric g_{AB}. It can be shown that, due to the presence of *a special- relativistic type of general covariance* (reparametrization invariances of the action under frame-preserving diffeomorphisms), the transition from a foliation to another one (i.e. a change of the notion of simultaneity) is a gauge transformation of the theory. Therefore, in parametrized Minkowski theories the *conventionalism of simultaneity* is rephrased as a *gauge problem* (in a way different from Refs.[18, 22]), i.e. as the *arbitrary choice of a gauge fixing selecting a well defined notion of simultaneity among those allowed by the gauge freedom*. Moreover, for every isolated system there is a preferred notion of simultaneity, namely the one associated with the 3+1 splitting whose leaves are the *Wigner hyper-planes* orthogonal to the conserved 4-momentum of the system: *this preferred simultaneity, intrinsically selected by the isolated system, leads to the Wigner-covariant rest-frame instant form of dynamics*.

The main property of *each foliation with simultaneity surfaces* associated to an admissible 4-coordinate transformation is that the embedding of the leaves of the foliation automatically determine two time-like vector fields and therefore *two congruences of (in general) non-inertial time-like observers*:

i) The time-like vector field $l^\mu(\tau, \vec{\sigma})$ of the normals to the simultaneity surfaces Σ_τ (by construction surface-forming, i.e. irrotational), whose flux lines are the world-lines of the so-called (in general non-inertial) Eulerian observers. The simultaneity surfaces Σ_τ are (in general non-flat) Riemannian 3-spaces in which the physical system is visualized and in each point the tangent space to Σ_τ is the local observer rest frame $R_{\tilde{l}(\tau_\gamma)}$ of the Eulerian observer through that point. This 3+1 viewpoint is called *hyper-surface 3+1 splitting*.

ii) The time-like evolution vector field $z^\mu_\tau(\tau, \vec{\sigma}) / \sqrt{\epsilon\, g_{\tau\tau}(\tau, \vec{\sigma})}$, which in general is not

surface-forming (i.e. it has non-zero vorticity like in the case of the rotating disk). The observers associated to its flux lines have the local observer rest frames $R_{\tilde{u}(\tau_\gamma)}$ not tangent to Σ_τ: there is no notion of 3-space for these observers (1+3 point of view or *threading splitting*) and no visualization of the physical system in large. However these observers can use the notion of simultaneity associated to the embedding $z^\mu(\tau, \vec{\sigma})$, which determines their 4-velocity. This 3+1 viewpoint is called *slicing 3+1 splitting*. In the case of the uniformly rotating disk all the existing rotating 4-coordiinate systems have a *coordinate singularity* (the horizon problem: $g_{oo}(y^o, \vec{y}) = 0$) where $\omega\, r = c$: there the time-like observers of the congruence would become null observers like on the horizon of a Schartzschild black hole and this is not acceptable in absence of a horizon.

As a consequence the plan of the paper is the following one.

In Section II we give some preliminaries on reference frames and on the notions of simultaneity in the 3+1 and 1+3 points of view with the associated method of synchronization of clocks and of definition of spatial distances.

In Section III we shall define the class of foliations implementing the idea behind the locality hypothesis that a non-inertial observer is equivalent to a continuous family of comoving inertial observers (as shown by Havas [23], there are other admissible foliations not in this class). Eqs.(1.1) will put restrictions on the comoving observers. The main byproduct of these restrictions will be that there exist admissible 4-coordinate transformations interpretable as *rigid systems of reference with arbitrary translational acceleration*. However there is *no admissible 4-coordinate transformation corresponding to a rigid system of reference with rotational motion*. When rotations are present, the admissible 4-coordinate transformations give rise to a continuum of local systems of reference like it happens in general relativity (*differential rotations*). Moreover our parametrization of this class of foliations uses an arbitrary centroid $x^\mu(\tau)$ with a time-like world-line as origin of the 3-coordinates on the simultaneity surfaces Σ_τ: *therefore these foliations describe possible notions of simultaneity for the arbitrary non-inertial observer $x^\mu(\tau)$.*

In Section IV we describe the simplest foliations of the previous class, whose simultaneity surfaces are space-like hyper-planes with differentially rotating coordinates.

In Section V we solve the following inverse problem: given a time-like unit vector field, i.e. a (in general not irrotational) congruence of non-inertial observers like that associated with a rotating disk, find an admissible foliation with simultaneity surfaces such that $z^\mu_\tau(\tau, \vec{\sigma})$ is

proportional to the given vector field.

Section VI contains some applications of our results. In Subsection A an operational method, generalizing Einstein's convention to arbitrary simultaneity surfaces, is proposed to build a grid of radar 4-coordinates to be used by a set of satellites of the GPS type. In Subsection B we give the 3+1 point of view on the rotating disk, while in Subsection C we give its description in the foliation of Section IV and we evaluate the Sagnac effect. In Subsection D the foliation of Section IV is used to describe Earth rotation (instead of assuming Einstein's convention plus Sagnac corrections as it happens in GPS) as an empirical admissible notion of simultaneity (see Subsection A) and it is applied to the determination of the one-way time transfer (including the Shapiro delay as a post-Newtonian effect) for the propagation of light from an Earth station to a satellite. Its comparison with the future measurements of the ACES mission will allow to synchronize the clocks according to this empirical simultaneity. In Subsection E we describe electro-magnetism as a parametrized Minkowski theory and we arrive at Maxwell equations in non-inertial frames as a result of gauge fixings determining an arbitrary admissible notion of simultaneity.

In the Conclusions we give a general overview of the results obtained and we discuss the *dynamical* nature of the admissible notions of simultaneity in general relativity.

In Appendix A there is a sketch of the derivation of the Sagnac effect from the non-inertial Maxwell equations.

In Appendix B there is the study of a family of admissible embeddings, whose leaves are parallel hyper-planes, closed under the action of the Lorentz group.

II. MORE ABOUT 3+1 VERSUS 1+3 NOTIONS OF SIMULTANEITY.

In this Section we shall collect some differential geometry tools needed to compare the 1+3 approximate notion of simultaneity with the exact 3+1 admissible ones. Due to the non-uniformity in the nomenclature used in the literature, let us first introduce some definitions following Ref.[39].

An *inertial observer* in Minkowski space-time M^4 is a time-like future-oriented straight line γ. Any point P on γ together with the unit time-like tangent vector $e^\mu_{(o)}$ to γ at P is an *instantaneous inertial observer*. Let us choose a point P on γ as the origin of an *inertial system* I_P having γ as time axis and three orthogonal space-like straight lines orthogonal to γ in P, with unit tangent vectors $e^\mu_{(r)}$, $r = 1,2,3$ as space axes. Let x^μ be a Cartesian 4-coordinate system referred to these axes, in which the line element has the form $ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$ with $\eta_{\mu\nu} = \boldsymbol{\epsilon}\,(+---)$, $\boldsymbol{\epsilon} = \pm 1$. Associated to these coordinates there is a *reference frame* (or *system of reference* or *platform* [2] [20]) given by the congruence of time-like straight lines parallel to γ, namely a unit vector field $u^\mu(x)$. Each of the integral lines of the vector field is identified by a fixed value of the three spatial coordinates x^i and represent an observer: this is a *reference point* according to Møller [12].

A reference frame l, i.e. a time-like vector field $l^\mu(x)\, \frac{\partial}{\partial x^\mu}$ with its congruence of time-like world-lines and its associated 1+3 splitting of TM^4, admits the decomposition [21]

$$
\begin{aligned}
D^{(\eta)}_\mu\, l_\nu \;&=\; l_\mu\, a_\nu + \frac{1}{3}\,\Theta\, P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu},\\
&a^\mu = l^\nu\, D^{(\eta)}_\nu\, l^\mu,\\
&\Theta = D^{(\eta)}_\mu\, l^\mu,\\
&\sigma_{\mu\nu} = \frac{1}{2}\,(a_\mu\, l_\nu + a_\nu\, l_\mu) + \frac{1}{2}\,(D^{(\eta)}_\mu\, l_\nu + D^{(\eta)}_\nu\, l_\mu) - \frac{1}{3}\,\Theta\, P_{\mu\nu},\\
&\textit{with magnitude } \sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu},\\
&\omega_{\mu\nu} = -\omega_{\nu\mu} = \epsilon_{\mu\nu\alpha\beta}\,\omega^\alpha\, l^\beta = \frac{1}{2}\,(a_\mu\, l_\nu - a_\nu\, l_\mu) + \frac{1}{2}\,(D^{(\eta)}_\mu\, l_\nu - D^{(\eta)}_\nu\, l_\mu),\\
&\omega^\mu = \frac{1}{2}\,\epsilon^{\mu\alpha\beta\gamma}\,\omega_{\alpha\beta}\, l_\gamma,
\end{aligned}
\tag{2.1}
$$

where a^μ is the 4-acceleration, Θ the *expansion* [22], $\sigma_{\mu\nu}$ the *shear* [23] and $\omega_{\mu\nu}$ the *twist*

[20] According to Ref.[2] a reference frame is a platform with a tetrad field defining a tetrad for each observer.

[21] $P_{\mu\nu}(x) = \eta_{\mu\nu} - \boldsymbol{\epsilon}\, l_\mu(x)\, l_\nu(x)$ is the 3-metric in the rest-frame in the point x^μ, i.e. in the tangent 3-plane orthogonal to $l^\mu(x)$; $D^{(\eta)}$ is the Levi-Civita covariant derivative on Minkowski space-time.

[22] It measures the average expansion of the infinitesimally nearby world-lines surrounding a given world-line

or vorticity or rotation [24]; $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are purely spatial ($\sigma_{\mu\nu} l^\nu = \omega_{\mu\nu} l^\nu = 0$). Due to the Frobenius theorem, the congruence is (locally) *hyper-surface orthogonal* if and only if $\omega_{\mu\nu} = 0$. The equation $\frac{1}{l}\, l^\mu\, \partial_\mu\, l = \frac{1}{3}\, \Theta$ defines a representative length l along the world-line of l^μ, describing the volume expansion (or contraction) behaviour of the congruence.

Another important characterization of a reference frame l is its *synchronizability*. If we define the 1-form $\alpha_l = \eta_{\mu\nu}\, l^\nu(x)\, dx^\mu = l_\mu(x)\, dx^\mu$, then the reference frame l is said to be [39]:

i) *locally synchronizable* iff $\alpha_l \wedge d\alpha_l = 0$, i.e. $d\alpha_l = \beta \wedge \alpha_l$;

ii) *locally proper time synchronizable* iff $d\alpha_l = 0$, i.e. $\alpha_l = d\beta_l$;

iii) *synchronizable* iff there are C^∞ functions h, t from M^4 to R such that $\alpha_l = h\, dt$ and $\boldsymbol{\epsilon}\, h > 0$;

iv) *proper time synchronizable* iff $\alpha_l = dt$.

Since we have $\alpha_l \wedge d\alpha_l = 0 \quad \Leftrightarrow \quad \omega_l = \omega_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu = 0$, rotating reference frames, i.e. with nonzero vorticity, are *not* locally synchronizable, i.e. these reference frames are not surface-forming (or in other words the 1+3 splitting does not imply a 3+1 splitting with simultaneity surfaces *orthogonal* to the integral lines in the congruence). Therefore to give a reference frame (a congruence) in general does not define a notion of simultaneity and an allowed coordinate transformation. In Section V, given a non-synchronizable reference frame $u^\mu(x)$, we will show how it is possible to find embeddings $z^\mu(\tau, \vec{\sigma})$ whose leaves Σ_τ [*not orthogonal* to the integral lines of $u^\mu(x)$] are possible notions of simultaneity related to the non-synchronizable reference frame.

An *inertial reference frame* is defined as a covariantly constant vector field I, i.e. $D^{(\eta)}\, I = 0$ [25] , and is identified as $(I_P, e^\mu_{(A)}, x^\mu)$, $A = 0, 1, 2, 3$ and for any point Q we have $Q - P =$

---

in the congruence.

[23] It measures how an initial sphere in the tangent space to the given world-line, which is Lie transported along l^μ (i.e. it has zero Lie derivative with respect to $l^\mu\, \partial_\mu$), is distorted towards an ellipsoid with principal axes given by the eigenvectors of $\sigma^\mu{}_\nu$, with rate given by the eigenvalues of $\sigma^\mu{}_\nu$.

[24] It measures the rotation of the nearby world-lines infinitesimally surrounding the given one.

[25] In general relativity there is no frame I satisfying $D^{(g)}\, I = 0$, so that inertial reference frames do not exist. As shown in Ref.[91] in general relativity we can define: i) *pseudo-inertial reference frames* Q, which have all the integral lines in free fall, are non-rotating and at least locally synchronizable [i.e. $D_Q\, Q = 0$ and $\alpha_Q \wedge d\alpha_Q = 0$, where $\alpha_Q = {}^4g_{\mu\nu}(x)\, Q^\mu(x)\, dx^\mu$ if $Q = Q^\mu(x)\, \frac{\partial}{\partial x^\mu}$]; in naturally adapted 4-coordinates we have $Q = \frac{\partial}{\partial y^o}$; ii) *local Lorentz reference frames* L associated to a geodetic line γ with associated *local Lorentz (or Riemann normal)* 4-coordinates, for which only γ is in free fall, while outside γ we have $D_L\, L \neq 0$ and $\alpha_L \wedge d\alpha_L \neq 0$ (not synchronizable); along the geodesic γ we can build a *Fermi-Walker*

$x_\mu\, e^\mu_{(A)} \in T_P M^4$ [26] . Other inertial systems with origin P are obtained from I_P with global rigid Lorentz transformations, while Poincare' transformations describe also a change of origin.

To each time-like observer with world-line $x^\mu(\tau)$ we can associate an orthonormal tetrad whose time axis is $E^\mu_\tau(\tau) = \frac{\dot x^\mu(\tau)}{\sqrt{\boldsymbol{\epsilon}\, \dot x^2(\tau)}}$. The spatial axes are realized with the rotation axes of three mutually orthogonal gyroscopes (see footnote 12). The orientation of the gyroscopes is a matter of convention. Two widely used conventions are:

α) the three axes always point the fixed stars (like in Gravity Probe B) and are not Fermi-Walker transported;

β) the three axes point the fixed stars at $\tau = 0$ and then are Fermi-Walker transported along $x^\mu(\tau)$ (standard of non-rotation).

About gyroscopes see Ref.[92] and its bibliography.

As said in the Introduction, according to Møller [12] the more general *admissible coordinate transformations* in special relativity are those transformations $x^\mu \mapsto y^\mu = f^\mu(x)$ [with inverse $x^\mu = h^\mu(y)$] such that the associated 4-metric $g_{\mu\nu}(y) = \frac{\partial h^\alpha(y)}{\partial y^\mu}\, \frac{\partial h^\beta(y)}{\partial y^\nu}\, \eta_{\alpha\beta}$ satisfies the conditions (1.1) and the asymptotic conditions at spatial infinity described after Eqs.(1.1). In this way we get a new reference frame: the time-like lines of the new congruence are identified by $y^i = f^i(x) = const$. Therefore *a reference-frame-preserving coordinate transformation* must be of the type $y^o = f^o(x^\mu)$, $y^i = f^i(x^k)$. In the new admissible coordinate system the simultaneity surfaces are determined by $y^o = f^o(x) = const.$: they are space-like hyper-surfaces, with the one $y^o = f^o(x) = 0$ passing through P.

As emphasized by Havas [23] the transition from a global inertial system, with its standard clock (showing the same proper-time rate when placed at rest at the same place and with the proper time equal to the Cartesian coordinate time since $\eta_{oo} = \boldsymbol{\epsilon}$ and with Einstein's synchronization of all the clocks with $x^o = ct = const.$), to arbitrary allowed 4-coordinates only means to use arbitrary curvilinear spatial 3-coordinates, non-standard clocks (with the proper time - coordinate time connection $s_{12} = c\, \mathcal{T}_{12} = \int_1^2 \sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}\, dy^o$), a non-standard definition of simultaneity ($dy^o = 0$ so that the space-like simultaneity surfaces are defined by $y^o = f^o(x) = const.$) and (*non-rigid*) arbitrary reference frames (defined by the time-like

transported inertial moving frame as a standard of no-rotation.

[26] The vector fields $e^\mu_{(A)}(x)$ form an orthonormal basis of the tangent spaces $T_P M^4$ and are called an *orthonormal moving frame*.

vector field $l^\mu(y)\,\frac{\partial}{\partial y^\mu}$ describing the field of unit normals to the simultaneity surfaces) as in general relativity as emphasized also by Fock [93].

Since the line element in the allowed 4-coordinates is $ds^2 = g_{\mu\nu}(y)\,dy^\mu\,dy^\nu$, in general with $g_{oi}(y) \neq 0$ (*non-time-orthogonal metric*), the determination of *spatial length* and the synchronization of clocks can be done in two different ways like in general relativity:

A) *globally*, but in a coordinate-dependent way, by using the $dy^o = 0$ exact notion of (coordinate) simultaneity, defined using Einstein's convention, associated to the chosen allowed 4-coordinates (the 3+1 point of view) and the related notion of instantaneous 3-space;

B) *locally*, but in a coordinate-independent, by means of light signals by using Einstein's convention to define local simultaneity (the 1+3 point of view), as done for instance in Landau-Lifschitz [11], but with the lacking notion of intantaneous 3-space replaced by the local rest frame of some observer.

Let us compare these two notions of simultaneity and their implications for the synchronization of clocks and for the definition of spatial distance between different points in Minkowski space-time [27].

A) All the events on the space-like hyper-surface Σ_{y^o}, $y^o = f^o(x) = const.$, are simultaneous, namely simultaneity is realized with the condition $dy^o = 0$ ($c\,d\mathcal{T} = \sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}\,dy^o = 0$) on the coordinate time. Therefore, there must be a generalization of Einstein's convention using light rays implying the possibility to realize the synchronization of all the clocks lying on the instantaneous 3-space $y^o(x) = const.$. We will see in Section VI, Subsection A, how to define such a generalization. Two simultaneous nearby events A and B will have 4-coordinates $(y^o; y^i_A)$ and $(y^o; y^i_B = y^i_A + dy^i)$ respectively and their (coordinate) spatial distance will be

$$dl_{AB} = \sqrt{-\boldsymbol{\epsilon}\, g_{ij}(y_A)\,dy^i\,dy^j}. \tag{2.2}$$

Let us remark that in each event the coordinate time increment is parallel to the normal $l^\mu(y)$ to the simultaneity surface through that event.

[27] Let us remember that in an inertial system with Cartesian 4-coordinates, all the clocks on the simultaneity surface $x^o = ct = const.$ are synchronized with Einstein's convention using light signals or, equivalently, by slow transport of clocks. This is equivalent to the standard statement of relativity books that the absolute chrono-geometrical structure of Minkowski space-time is realized by putting a clock and rods in each point with the clocks on the hyper-planes $x^o = const.$ synchronized.

If the simultaneous events A and B are at finite distance on σ_{y^o}, to each space-like path $\mathcal{P}$ joining them is associated a distance $L^{(\mathcal{P})}_{AB} = \int_{\mathcal{P}} dl_{AB}$, as said in footnote 1, which is extremized by choosing the 3-geodesics joining them on Σ_{y^o}.

B) The usual strategy to define local simultaneity in general relativity (where it works only locally for nearby pairs of events due to the gravitational field [28]), as exemplified in Ref.[11], is the Martzke-Wheeler extension [24], adapted to accelerated observers, of Einstein's convention using test light rays (see for instance Ref.[95]).

Given an event A with 4-coordinates $y^\mu = f^\mu(x)$ and proper time

$$\mathcal{T}_A = \frac{1}{c} \int_o^A \sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}\, dy^o, \tag{2.3}$$

a nearby (locally) simultaneous event B will have 4-coordinates $y^\mu_B = y^\mu + \Delta\, y^\mu$ with $\Delta y^o \neq 0$ determined by Einstein's convention in the following way.

If A receives a light signal from a nearby event B_- of 4-coordinates $y^\mu_- = y^\mu + \delta\, y^\mu_- = (y^o + \delta\, y^o_-; y^i + \Delta y^i)$ with $\delta y^o_- < 0$, then δy^o_- is determined [29] by the condition $ds^2 = 0$ with the result

$$\begin{aligned}
\delta\, y^o_- &= \frac{1}{g_{oo}(y)} \Big(- g_{oi}(y)\, \Delta y^i - \boldsymbol{\epsilon} \sqrt{[g_{oi}(y)\, g_{oj}(y) - g_{oo}(y)\, g_{ij}(y)]\, \Delta y^i\, \Delta y^j} \Big) = \\
&= - \frac{g_{oi}(y)}{g_{oo}(y)}\, \Delta y^i - \frac{\boldsymbol{\epsilon}}{\sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}} \sqrt{-\boldsymbol{\epsilon} \Big(g_{ij}(y) - \frac{g_{oi}(y)\, g_{oj}(y)}{g_{oo}(y)} \Big) \Delta y^i\, \Delta y^j} < 0.
\end{aligned} \tag{2.4}$$

Then A re-transmits the signal to a nearby event B_+, where the light signal is absorbed, of 4-coordinates $y^\mu_+ = y^\mu + \delta\, y^\mu_+ = (y^o + \delta\, y^o_+; y^i + \Delta\, y^i)$ with $\delta y^o_+ > 0$. Now $ds^2 = 0$ gives the following expression for δy^o_+

$$\begin{aligned}
\delta\, y^o_+ &= \frac{1}{g_{oo}(y)} \Big(- g_{oi}(y)\, \Delta y^i + \boldsymbol{\epsilon} \sqrt{[g_{oi}(y)\, g_{oj}(y) - g_{oo}(y)\, g_{ij}(y)]\, \Delta y^i\, \Delta y^j} \Big) = \\
&= - \frac{g_{oi}(y)}{g_{oo}(y)}\, \Delta y^i + \frac{\boldsymbol{\epsilon}}{\sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}} \sqrt{-\boldsymbol{\epsilon} \Big(g_{ij}(y) - \frac{g_{oi}(y)\, g_{oj}(y)}{g_{oo}(y)} \Big) \Delta y^i\, \Delta y^j} > 0.
\end{aligned} \tag{2.5}$$

[28] But for globally hyperbolic space-times the global simultaneity A) of the 3+1 point of view can be defined and is used [88, 94] in canonical ADM [87] metric gravity.

[29] The two solutions of $ds^2 = 0$ are $\hat{\delta} y^o_{(\pm)} = - \frac{g_{oi}(y)}{g_{oo}(y)}\, \Delta y^i \pm \frac{1}{g_{oo}(y)} \sqrt{[g_{oi}(y)\, g_{oj}(y) - g_{oo}(y)\, g_{ij}(y)]\, \Delta y^i\, \Delta y^j}$. With our conventions on the signature of the metric ($\boldsymbol{\epsilon}\, g_{oo}(y) > 0$, $-\boldsymbol{\epsilon}\, g_{ij}(y) > 0$), for $\boldsymbol{\epsilon} = +$ we get $\hat{\delta} y^o_{(+)} > 0$ and $\hat{\delta} y^o_{(-)} < 0$, while for $\boldsymbol{\epsilon} = -$ we get $\hat{\delta} y^o_{(-)} > 0$ and $\hat{\delta} y^o_{(+)} < 0$. This implies Eq.(2.4).

Then the coordinate time $y^o_B = y^o + \Delta\, y^o$ of the *(locally) simultaneous event* B (in between events B_- and B_+) is determined by the *Einstein convention*

$$\Delta\, y^o \stackrel{def}{=} \frac{1}{2}\,(\delta\, y^o_+ + \delta\, y^o_-) = -\frac{g_{oi}(y)}{g_{oo}(y)}\,\Delta\, y^i. \tag{2.6}$$

This corresponds to replace the exact global simultaneity definition $dy^o = 0$ (or $d\mathcal{T} = \frac{1}{c}\,\sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}\, dy^o = 0$) given in A) with the local approximate one implied by the vanishing of the so-called *Einstein synchronized proper-time pseudo-interval*

$$c\,\Delta\,\widetilde{\mathcal{T}} = \sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}\,\Delta\, y^o + \frac{\boldsymbol{\epsilon}\, g_{oi}(y)}{\sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}}\,\Delta\, y^i \stackrel{def}{=} \sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}\,\Delta\, y^o + \Delta_{y^o} = 0, \tag{2.7}$$

which is not proportional to an exact 1-form (it is not an interval) like it happens for $c\,d\,\mathcal{T} = 0$. Therefore, with this definition of simultaneity, the statement that two nearby events A and B are (locally) simultaneous requires the use of two events B_- and B_+ (in the past and in the future of B respectively) with the following difference of coordinate time

$$\begin{aligned}
\delta\, y^o \;&=\; \delta\, y^o_+ - \delta\, y^o_- = \frac{2\,\boldsymbol{\epsilon}}{g_{oo}(y)}\,\sqrt{[g_{oi}(y)\, g_{oj}(y) - g_{oo}(y)\, g_{ij}(y)]\,\Delta y^i\,\Delta y^j} = \\
&=\; \frac{2}{\sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}}\,\sqrt{-\boldsymbol{\epsilon}\,\Big(g_{ij}(y) - \frac{g_{oi}(y)\, g_{oj}(y)}{g_{oo}(y)}\Big)\,\Delta y^i\,\Delta y^j}.
\end{aligned} \tag{2.8}$$

Since we can re-write the line element $ds^2 = g_{\mu\nu}(y)\, dy^\mu\, dy^\nu$ as

$$ds^2 = \boldsymbol{\epsilon}\,\Big(\sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}\, dy^o + \frac{\boldsymbol{\epsilon}\, g_{oi}(y)}{\sqrt{\boldsymbol{\epsilon}\, g_{oo}(y)}}\, dy^i\Big)^2 + \Big(g_{ij}(y) - \frac{g_{oi}(y)\, g_{oj}(y)}{g_{oo}(y)}\Big)\, dy^i\, dy^j, \tag{2.9}$$

the *(pseudo-proper) spatial distance* between the (locally) simultaneous events A and B implied by the local simultaneity notion $c\,\Delta\,\widetilde{\mathcal{T}} = 0$ is

$$\Delta l_{AB} = \sqrt{-\boldsymbol{\epsilon}\,\Big(g_{ij}(y) - \frac{g_{oi}(y)\, g_{oj}(y)}{g_{oo}(y)}\Big)\,\Delta y^i\,\Delta y^j} \stackrel{def}{=} \sqrt{{}^3\gamma_{ij}(y)\,\Delta y^i\,\Delta y^j}. \tag{2.10}$$

This justifies the use of the spatial 3-metric ${}^3\gamma_{ij}(y) = -\boldsymbol{\epsilon}\,[g_{ij}(y) - \frac{g_{oi}(y)\, g_{oj}(y)}{g_{oo}(y)}]$ of signature $(+++)$, which satisfies the analogue of the spatial conditions of Eq.(1.1) [11].

In each point, remembering $y^\mu = f^\mu(x)$ with inverse $x^\mu = h^\mu(y)$, we have

$$
\begin{aligned}
c\,\Delta\,\widetilde{\mathcal{T}} \;&=\; \frac{\epsilon}{\sqrt{\epsilon\, g_{oo}(y)}}\, \frac{\partial h^\alpha(y)}{\partial y^o}\, \eta_{\alpha\beta} \Big[\frac{\partial h^\beta(y)}{\partial y^o}\, \Delta y^o + \frac{\partial h^\beta(y)}{\partial y^i}\, \Delta y^i\Big] = \\
&=\; \frac{\epsilon}{\sqrt{\epsilon\, g_{oo}(y)}}\, \frac{\partial h_\alpha(y)}{\partial y^o}\, \Delta x^\alpha \overset{def}{=} \epsilon\, u_\mu(y)\, \Delta y^\mu.
\end{aligned}
\tag{2.11}
$$

The time-like vector field $u^\mu(y) = \frac{1}{\sqrt{\epsilon\, g_{oo}(y)}} \frac{\partial h^\mu(y)}{\partial y^o}$ defines a (in general non-surface-forming) reference frame (1+3 point of view), which in each point is orthogonal to the space-like vector

$$
\Delta l^\mu_{AB\perp} = \Big(\frac{\partial h^\mu(y)}{\partial y^i} - \frac{g_{oi}(y)}{g_{oo}(y)}\, \frac{\partial h^\mu(y)}{\partial y^o}\Big)\, \Delta y^i = [\delta^\mu_\nu - \epsilon\, u^\mu(y)\, u_\nu(y)]\, \Delta x^\nu,
\tag{2.12}
$$

lying in the rest frame tangent plane at A in $T_A M^4$ (the local rest frame R_u). The spatial distance can be re-written as [30] $(\Delta l_{AB})^2 = -\epsilon\, \Delta l^\mu_{AB\perp}\, \eta_{\mu\nu}\, \Delta l^\nu_{AB\perp} = \frac{\epsilon\, g_{oo}(y)}{4}\, (\delta y^o)^2 = \frac{\epsilon\, g_{oo}(y)}{4}\, [\delta y^o_+ - \delta y^o_-]^2$. This shows that we can define the line pseudo-element with the coordinate-independent time-orthogonal decomposition

$$
(\Delta s_{AB})^2 = \epsilon \Big[c^2\, (\Delta \tilde{\mathcal{T}}_{AB})^2 - (\Delta l_{AB})^2\Big].
\tag{2.13}
$$

However, since the local observer rest frame R_u is not tangent to an instantaneous 3-space, we cannot define the spatial distance of non-nearby simultaneous events. In phenomenological calculations R_u is identified with the instantaneous 3-space of the observer (see footnote 11) and the finite spatial distance of a point P from the observer is defined by considering the 3-geodesic joining P to the observer.

As said in the Introduction, instead of the 4-coordinates y^μ, it is convenient to use the Lorentz scalar (radar) 4-coordinates $\sigma^A = (\tau, \vec{\sigma}) = \sigma^A(x)$ and the embedding (inverse coordinate transformation) $x^\mu = z^\mu(\sigma) = z^\mu(\tau, \vec{\sigma})$, naturally adapted to the simultaneity leaves Σ_τ of the 3+1 splitting. In these 4-coordinates the previous two cases A) and B) are re-formulated in the following way

[30] This equation give us a expression of the metric ${}^3\gamma_{ij}(y)$ in terms of δy^o_+, δy^o_-. Instead from Eqs.(2.5),(2.6) we get the following expression [28] (Chapter III, Section 40) for dl_{AB} of Eq.(2.2)

$$
(dl_{AB})^2 = -\epsilon\, g_{ij}(y)\, \Delta y^i\, \Delta y^j = -\epsilon\, g_{oo}(y)\, \delta y^o_+ \cdot \delta y^o_- .
$$

A) We use the synchronization $l_\mu(\sigma)\, dz^\mu(\sigma) = N\, d\tau = 0$ (synchronizzability of the reference frame of normals to the simultaneity surfaces), whose associated spatial distance of simultaneous events is

$$dl_{AB} = \sqrt{-\boldsymbol{\epsilon}\, g_{rs}(\tau, \vec{\sigma})\, d\sigma^r\, d\sigma^s}. \tag{2.14}$$

B) We use the time-like 4-vector

$$u^\mu(\tau, \vec{\sigma}) = \frac{z^\mu_\tau(\tau, \vec{\sigma})}{\sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\sigma})}}, \tag{2.15}$$

which defines a reference frame, in general non-synchronizable, and we use the local notion of simultaneity ($\Delta\tau_E$ is an Einstein synchronized non-proper-time pseudo-interval)

$$\begin{aligned} c\, \Delta\widetilde{\mathcal{T}} &= \boldsymbol{\epsilon}\, u_\mu(\sigma)\, \Delta z^\mu(\sigma) \stackrel{def}{=} \sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\sigma)}\, \Delta\tau + \Delta_\tau = \\ &= \frac{\boldsymbol{\epsilon}\, z_{\tau\mu}(\tau, \vec{\sigma})}{\sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\sigma})}} \Big[z^\mu_\tau(\tau, \vec{\sigma})\, \Delta\tau + z^\mu_r(\tau, \vec{\sigma})\, \Delta\sigma^r \Big] = \\ &= \sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\sigma})}\, \Delta\tau + \frac{g_{\tau r}(\tau, \vec{\sigma})}{\sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\sigma})}}\, \Delta\sigma^r \stackrel{def}{=} \sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\sigma)}\, \Delta\tau_E = 0. \end{aligned} \tag{2.16}$$

Now we get the solutions $\delta\tau_\pm = -\frac{g_{\tau r}}{g_{\tau\tau}}\, \Delta\sigma^r \pm \frac{\boldsymbol{\epsilon}}{\sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}}} \sqrt{-\epsilon\, \Big(g_{rs} - \frac{g_{\tau r}\, g_{\tau s}}{g_{\tau\tau}} \Big)\, \Delta\sigma^r\, \Delta\sigma^s}$ ($\delta\tau_+ > 0$, $\delta\tau_- < 0$) of $ds^2 = 0$, so that the Einstein convention becomes

$$\Delta\tau \stackrel{def}{=} \frac{1}{2}\, (\delta\,\tau_+ + \delta\,\tau_-) = -\frac{g_{\tau r}(\tau, \vec{\sigma})}{g_{\tau\tau}(\tau, \vec{\sigma})}\, \Delta\sigma^r. \tag{2.17}$$

By defining

$$\Delta l^\mu_{AB\perp} = \Big(z^\mu_r - \frac{g_{\tau r}}{g_{\tau\tau}}\, z^\mu_\tau \Big)\, \Delta\sigma^r, \tag{2.18}$$

we arrive at the following definition of *pseudo spatial distance* ($\delta\tau = \delta\,\tau_+ - \delta\,\tau_-$) [31]

$$\begin{aligned} \Delta l^2_{AB\perp} &= -\boldsymbol{\epsilon}\, \Big(g_{rs}(\tau, \vec{\sigma}) - \frac{g_{\tau r}(\tau, \vec{\sigma})\, g_{\tau s}(\tau, \vec{\sigma})}{g_{\tau\tau}(\tau, \vec{\sigma})} \Big)\, \Delta\sigma^r\, \Delta\sigma^s = \\ &= {}^3\gamma_{rs}(\tau, \vec{\sigma})\, \Delta\,\sigma^r\, \Delta\,\sigma^s = \frac{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\sigma})}{4}\, (\delta\tau)^2, \end{aligned} \tag{2.19}$$

[31] Moreover we have $(dl_{AB})^2 = -\boldsymbol{\epsilon}\, g_{rs}(\tau, \vec{\sigma})\, d\sigma^r\, d\sigma^s = -\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\sigma})\, \delta\tau_+\, \delta\tau_-$.

with the 3-metric ${}^3\gamma_{rs}$ of signature $(+++)$.

Let us define the *one-way velocity of a ray of light* according to the two points of view A) and B) on simultaneity. A ray of light has a null geodesic $z^\mu(\lambda) = z^\mu(\tau(\lambda), \vec{\sigma}(\lambda))$ [λ affine parameter along the null geodesic] of Minkowski space-time as world-line, defined by the condition [use Eq.(2.9)]

$$
\begin{aligned}
&ds^2 = \\
&= \Big[g_{\tau\tau}(\sigma(\lambda)) \left(\frac{d\tau(\lambda)}{d\lambda^2}\right)^2 + 2\, g_{\tau r}(\sigma(\lambda))\, \frac{d\tau(\lambda)}{d\lambda}\, \frac{d\sigma^r(\lambda)}{d\lambda} + g_{rs}(\sigma(\lambda))\, \frac{d\sigma^r(\lambda)}{d\lambda}\, \frac{d\sigma^s(\lambda)}{d\lambda} \Big]\, (d\lambda)^2 = \\
&= \Big[\boldsymbol{\epsilon} \left(\sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))} + \frac{\boldsymbol{\epsilon}\, g_{\tau r}(\tau, \vec{\eta}(\tau))}{\sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}}\, \frac{d\eta^r(\tau)}{d\tau} \right)^2 + \\
&- \boldsymbol{\epsilon}\ {}^3\gamma_{rs}(\tau, \vec{\eta}(\tau))\, \frac{d\eta^r(\tau)}{d\tau}\, \frac{d\eta^s(\tau)}{d\tau} \Big] \left(\frac{d\tau(\lambda)}{d\lambda}\right)^2 (d\lambda)^2 = 0,
\end{aligned}
\tag{2.20}
$$

where we have introduced the 3-coordinates $\vec{\eta}(\tau)$ for the light ray by means of $z^\mu(\tau(\lambda), \vec{\sigma}(\lambda)) = z^\mu(\tau(\lambda), \vec{\eta}(\tau(\lambda)))$. This condition implies

$$
g_{rs}(\tau, \vec{\eta}(\tau))\, \frac{d\eta^r(\tau)}{d\tau}\, \frac{d\eta^s(\tau)}{d\tau} + 2\, g_{\tau r}(\tau, \vec{\eta}(\tau))\, \frac{d\eta^r(\tau)}{d\tau} + g_{\tau\tau}(\tau, \vec{\eta}(\tau)) = 0 \tag{2.21}
$$

or

$$
{}^3\gamma_{rs}(\tau, \vec{\eta}(\tau))\, \frac{d\eta^r(\tau)}{d\tau}\, \frac{d\eta^s(\tau)}{d\tau} - \left(\sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))} + \frac{\boldsymbol{\epsilon}\, g_{\tau r}(\tau, \vec{\eta}(\tau))}{\sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}}\, \frac{d\eta^r(\tau)}{d\tau} \right)^2 = 0. \tag{2.22}
$$

The solution of this equation defines a different *one-way direction-dependent coordinate velocity of light* according to the points of view A) and B): $\frac{d\eta^r(\tau)}{d\tau} = \frac{d\eta(\tau)}{d\tau}\, \hat{n}^r$ with the 3-direction of propagation $\hat{n}$ normalized to the unity according to the chosen A) or B) notion of spatial distance.

A) We use the definition of one-way velocity $v_A = \frac{dl_{AB}}{d\tau}$ implied by Eq.(2.14). We put $-\boldsymbol{\epsilon}\, g_{rs}(\tau, \vec{\eta}(\tau))\, \hat{n}^r_A\, \hat{n}^s_A = 1$. Eq.(2.21) and its (future-pointing) solution are

$$\Big(\frac{d\eta_A(\tau)}{d\tau}\Big)^2 - 2\,\epsilon\, g_{\tau r}(\tau, \vec{\eta}(\tau))\, \hat{n}^r_A\, \frac{d\eta_A(\tau)}{d\tau} - \epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau)) = 0,$$

$$\begin{aligned}
v^r_A(\tau) &= \frac{d\eta_A(\tau)}{d\tau}\, \hat{n}^r_A = \Big[\epsilon\, g_{\tau s}(\tau, \vec{\eta}(\tau))\, \hat{n}^s_A + \sqrt{[\, g_{\tau s}(\tau, \vec{\eta}(\tau))\, \hat{n}^s_A\,]^2 + \epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}\,\Big]\, \hat{n}^r_A = \\
&= \Big[\frac{\epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}{-\epsilon\, g_{\tau s}(\tau, \vec{\eta}(\tau))\, \hat{n}^s_A + \sqrt{[g_{\tau s}(\tau, \vec{\eta}(\tau))\, \hat{n}^s_A]^2 + \epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}}\Big]\, \hat{n}^r_A, \\
\tilde{v}_A(\tau) &= \frac{c}{\sqrt{\epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}}\, \frac{d\eta_A(\tau)}{d\tau} = \\
&= c\, \frac{\epsilon\, g_{\tau s}(\tau, \vec{\eta}(\tau))\, \hat{n}^s_A + \sqrt{(g_{\tau s}(\tau, \vec{\eta}(\tau))\, \hat{n}^s_A)^2 + \epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}}{\sqrt{\epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}} \\
&\rightarrow\; c \qquad \text{if } g_{\tau r} = 0.
\end{aligned} \tag{2.23}$$

B) We use the definition of one-way velocity $v_B = \frac{\Delta l_{AB}}{d\tau}$ implied by Eq.(2.19). We put ${}^3\gamma_{rs}(\tau, \vec{\eta}(\tau))\, \hat{n}^r_B\, \hat{n}^s_B = 1$. Eq.(2.22) and its (future-pointing) solution are (see Refs. [12, 23])

$$\Big(\frac{d\eta_B(\tau)}{d\tau}\Big)^2 - \Big(\sqrt{\epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))} + \frac{\epsilon\, g_{\tau r}(\tau, \vec{\eta}(\tau))}{\sqrt{\epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}}\, \hat{n}_B\, \frac{d\eta_B(\tau)}{d\tau}\Big)^2 = 0,$$

$$\begin{aligned}
v^r_B(\tau) &= \frac{d\eta_B(\tau)}{d\tau}\, \hat{n}^r_B = \frac{g_{\tau\tau}(\tau, \vec{\eta}(\tau))}{\sqrt{\epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))} - \epsilon\, g_{\tau s}(\tau, \vec{\eta}(\tau))\, \hat{n}^s_B}\, \hat{n}^r_B, \\
\tilde{v}_B(\tau) &= \frac{c}{\sqrt{\epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}}\, \frac{d\eta_B(\tau)}{d\tau} = c\, \frac{\sqrt{\epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))}}{\sqrt{\epsilon\, g_{\tau\tau}(\tau, \vec{\eta}(\tau))} - \epsilon\, g_{\tau s}(\tau, \vec{\eta}(\tau))\, \hat{n}^s_B} \\
&\rightarrow\; c \qquad \text{if } g_{\tau r} = 0.
\end{aligned} \tag{2.24}$$

Havas [23] showed that, if we use admissible coordinate transformations $y^\mu = f^\mu(x)$ instead of adapted coordinates σ^A, then the analogue of Eq.(2.24) corresponds to a generalization of Einstein's convention of the type $x^o_A = x^o_{B_-} + E\, (x^o_{B_+} - x^o_{B_-})$, $0 < E < 1$ with a position- and direction- dependent $E = \frac{x^o_{B_-} - x^o_A}{x^o_{B_+} - x^o_{B_-}}$ in place of the constant E of footnote 4. Instead in adapted coordinates we get $\tau_A = \tau_{B_-} + \mathcal{E}\, (\tau_{B_+} - \tau_{B_-})$, $0 < \mathcal{E} < 1$. See also Ref.[21] for the study of this type of non-standard synchrony and of the associated 3-metric to be used for measuring spatial distances.

Finally the local notion B) of simultaneity and the definition (2.19) of spatial distance imply the following null pseudo-interval [see Eq.(2.13)]

$$(\Delta s_{AB})^2 = \boldsymbol{\epsilon}\,[c^2\,(\Delta\widetilde{\mathcal{T}}_{AB})^2 - (\Delta l_{AB})^2] = 0, \tag{2.25}$$

and allow to define an invariant isotropic one-way pseudo-velocity

$$v_{AB} = \frac{\Delta l_{AB}}{\Delta\widetilde{\mathcal{T}}_{AB}} = c, \tag{2.26}$$

even for non-time-orthogonal metrics. This is a formal answer to Klauber's criticism [33], but, since $(\Delta\tilde{\mathcal{T}}_{AB})$ is neither the proper time of A nor that of B, the interpretation of this pseudo-velocity is not clear.

III. ADMISSIBLE 4-COORDINATES AND THE LOCALITY HYPOTHESIS: NON-EXISTENCE OF RIGID ROTATING REFERENCE FRAMES.

In Ref.[23] Havas proposed the following two examples (the second one is a time-dependent transformation) of simultaneity foliations associated with admissible 4-coordinates, i.e. whose 4-metric satisfies Eq.(1.1):

1)

$$y^o = x^o + \frac{1}{c^2}\, f(x^i), \qquad y^i = x^i, \qquad \text{with inverse } x^o = y^o - \frac{1}{c^2}\, f(y^i), \quad x^i = y^i,$$

and with associated 4-metric

$$g_{oo}(y) = -c^2, \quad g_{oi}(y) = \frac{\partial f(y)}{\partial y^i} \;(\text{ if } g_{oi} = const.,\ \text{then } (g_{oi})^2 < c^2),$$

$$g_{ij}(y) = \delta_{ij} - \frac{1}{c^2}\, \frac{\partial f}{\partial y^i}\, \frac{\partial f}{\partial y^j}; \tag{3.1}$$

2)

$$y^o = x^o\, f^{-1}(x^i), \qquad y^i = x^i, \qquad \text{with inverse } x^o = y^o\, f(y^i), \quad x^i = y^i,$$

and with associated 4-metric

$$g_{oo}(y) = -c^2\, f^2, \qquad g_{oi}(y) = -y^o\, c^2\, f\, \frac{\partial f}{\partial y^i}, \qquad g_{ij}(y) = \delta_{ij} - (y^o)^2\, c^2\, \frac{\partial f}{\partial y^i}\, \frac{\partial f}{\partial y^j}. \tag{3.2}$$

Both of them are examples of admissible 4-coordinate systems not interpretable in terms of comoving observers as required by the locality hypothesis.

Let us now consider a class of 4-coordinate transformations which implements the idea of accelerated observers as sequences of comoving observers (the locality hypothesis) and let us determine the conditions on the transformations to get a set of admissible 4-coordinates. From now on we shall use Lorentz-scalar radar-like 4-coordinates $\sigma^A = (\tau; \vec{\sigma})$ adapted to the foliation, whose simultaneity leaves are denoted Σ_τ.

As we have said, the admissible embeddings $x^\mu = z^\mu(\tau, \vec{\sigma})$ [inverse transformations of $x^\mu \mapsto \sigma^A(x)$], defined with respect to a given inertial system, must tend to parallel space-like hyper-planes at spatial infinity. If $l^\mu = l^\mu_{(\infty)} = \epsilon^\mu_\tau$, $l^2_{(\infty)} = \boldsymbol{\epsilon}$, is the asymptotic normal, let us define the asymptotic orthonormal tetrad ϵ^μ_A, $A = \tau, 1, 2, 3$, by using the standard Wigner boost for time-like Poincare' orbits $L^\mu{}_\nu(l_{(\infty)}, \overset{\circ}{l}_{(\infty)})$, $\overset{\circ}{l}_{(\infty)} = (1; \vec{0})$: $\epsilon^\mu_A \overset{def}{=} L^\mu{}_A(l_{(\infty)}, \overset{\circ}{l}_{(\infty)})$, $\eta_{AB} = \epsilon^\mu_A\, \eta_{\mu\nu}\, \epsilon^\nu_B$.

Then a parametrization of the asymptotic hyper-planes is $z^\mu = x^\mu_o + \epsilon^\mu_A\, \sigma^A = x^\mu(\tau) + \epsilon^\mu_r\, \sigma^r$ with $x^\mu(\tau) = x^\mu_o + \epsilon^\mu_\tau\, \tau$ a time-like straight-line (an asymptotic inertial observer). Let us define a family of 3+1 splittings of Minkowski space-time by means of the following embeddings

$$
\begin{aligned}
z^\mu(\tau, \vec{\sigma}) \;&=\; x^\mu_o + \Lambda^\mu{}_\nu(\tau, \vec{\sigma})\, \epsilon^\nu_A\, \sigma^A \;= \tilde{x}^\mu(\tau) + F^\mu(\tau, \vec{\sigma}),\\[2ex]
&\tilde{x}^\mu(\tau) = x^\mu_o + \Lambda^\mu{}_\nu(\tau, \vec{0})\, \epsilon^\nu_\tau\, \tau,\\
&F^\mu(\tau, \vec{\sigma}) = [\Lambda^\mu{}_\nu(\tau, \vec{\sigma}) - \Lambda^\mu{}_\nu(\tau, \vec{0})]\, \epsilon^\nu_\tau\, \tau + \Lambda^\mu{}_\nu(\tau, \vec{\sigma})\, \epsilon^\nu_r\, \sigma^r,\\[2ex]
&\Lambda^\mu{}_\nu(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} \delta^\mu_\nu, \quad \Rightarrow \;\; z^\mu(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} x^\mu_o + \epsilon^\mu_A\, \sigma^A = x^\mu(\tau) + \epsilon^\mu_r\, \sigma^r,
\end{aligned}
\tag{3.3}
$$

where $\Lambda^\mu{}_\nu(\tau, \vec{\sigma})$ are Lorentz transformations ($\Lambda^\mu{}_\alpha\, \eta_{\mu\nu}\, \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$) belonging to the component connected with the identity of $SO(3,1)$. While the functions $F^\mu(\tau, \vec{\sigma})$ determine the form of the simultaneity surfaces Σ_τ, the centroid $\tilde{x}^\mu(\tau)$, corresponding to an arbitrary time-like observer chosen as origin of the 3-coordinates on each Σ_τ, determines how these surfaces are packed in the foliation.

A variant are the embeddings

$$z^\mu(\tau, \vec{\sigma}) = x^\mu_o(\tau) + \Lambda^\mu{}_\nu(\tau, \vec{\sigma})\, \epsilon^\nu_A\, \sigma^A \rightarrow_{|\vec{\sigma}| \rightarrow \infty} x^\mu_o(\tau) + \epsilon^\mu_\tau\, \tau + \epsilon^\mu_r\, \sigma^r = x^\mu(\tau) + \epsilon^\mu_r\, \sigma^r, \qquad (3.4)$$

with $x^\mu(\tau) = x^\mu_o(\tau) + \epsilon^\mu_\tau\, \tau = \epsilon^\mu_\tau\, [\tau + f(\tau)]$ an arbitrary time-like straight-line (an inertial observer) not parametrized in terms of the proper time. The only difference now is that the asymptotic hyper-planes are no more uniformly spaced like in the case $x^\mu(\tau) = x^\mu_o + \epsilon^\mu_\tau\, \tau$ ($z^\mu_\tau = \epsilon^\mu_\tau \mapsto z^\mu_\tau(\tau) = \epsilon^\mu_\tau\, [1 + \dot{f}(\tau)]$).

Since the asymptotic foliation with parallel hyper-planes, having a constant vector field $l^\mu = \epsilon^\mu_\tau$ of normals, defines an inertial reference frame, we see that the foliation (3.3) with its associated non-inertial reference frame is obtained from the asymptotic inertial frame by means of *point-dependent Lorentz transformations*. As a consequence, the integral lines, i.e. the non-inertial observers and (non-rigid) non-inertial reference frames associated to this special family of simultaneity notions, are parametrized as a continuum of comoving inertial observers as required by the locality hypothesis [32].

Let us remark that when an arbitrary isolated system is described by a Minkowski parametrized theory, in which the embeddings $z^\mu(\tau, \vec{\sigma})$ are *gauge* configuration variables, the transition from the description of dynamics in one of these non-inertial reference frames compatible with the locality hypothesis to another arbitrary allowed reference frame, like the one of Eqs.(3.1), is a *gauge transformation*: *therefore in this case the locality hypothesis can always be assumed valid modulo gauge transformations.*

An equivalent parametrization of the embeddings of this family of reference frames is

$$z^\mu(\tau, \vec{\sigma}) \;=\; x^\mu_o + \epsilon^\mu_B\, \Lambda^B{}_A(\tau, \vec{\sigma})\, \sigma^A = x^\mu_o + U^\mu_A(\tau, \vec{\sigma})\, \sigma^A = \tilde{x}^\mu(\tau) + F^\mu(\tau, \vec{\sigma}),$$

$$\begin{aligned} \tilde{x}^\mu(\tau) &= x^\mu_o + U^\mu_\tau(\tau, \vec{0})\, \tau, \\ F^\mu(\tau, \vec{\sigma}) &= [U^\mu_\tau(\tau, \vec{\sigma}) - U^\mu_\tau(\tau, \vec{0})]\, \tau + U^\mu_r(\tau, \vec{\sigma})\, \sigma^r, \end{aligned} \qquad (3.5)$$

[32] The second non-inertial and non-surface-forming reference frame (the skew one) associated with these embeddings, with vector field $z^\mu_\tau(\tau, \vec{\sigma}) / \sqrt{\epsilon\, g_{\tau\tau}(\tau, \vec{\sigma})}$, asymptotically tends to the same asymptotic inertial reference frame because $z^\mu_\tau(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} \epsilon^\mu_\tau = l^\mu_{(\infty)}$ if $\partial_\tau\, \Lambda^\mu{}_\nu(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} 0$.

where we have defined:

$$\Lambda^B{}_A(\tau,\vec{\sigma}) \;=\; \epsilon^B_\mu\, \Lambda^\mu{}_\nu(\tau,\vec{\sigma})\, \epsilon^\nu_A, \qquad U^\mu_A(\tau,\vec{\sigma})\, \eta_{\mu\nu}\, U^\nu_B(\tau,\vec{\sigma}) = \epsilon^\mu_A\, \eta_{\mu\nu}\, \epsilon^\nu_B = \eta_{AB},$$

$$U^\mu_A(\tau,\vec{\sigma}) \;=\; \epsilon^\mu_B\, \Lambda^B{}_A(\tau,\vec{\sigma}) \rightarrow_{|\vec{\sigma}|\rightarrow\infty} \epsilon^\mu_A, \tag{3.6}$$

where $\epsilon^B_\mu = \eta_{\mu\nu}\, \eta^{BA}\, \epsilon^\nu_A$ are the inverse tetrads.

A slight generalization of these embeddings allows to find Nelson's [49] 4-coordinate transformation [but extended from $\vec{\sigma}$-independent Lorentz transformations $\Lambda^\mu{}_\nu = \Lambda^\mu{}_\nu(\tau)$ to $\vec{\sigma}$-dependent ones!] implying Møller rotating 4-metric [33]

$$z^\mu(\tau,\vec{\sigma}) \;=\; x^\mu_o + \epsilon^\mu_A \Big[\Lambda^A{}_B(\tau,\vec{\sigma})\, \sigma^B + V^A(\tau,\vec{\sigma})\Big],$$

$$V^\tau(\tau,\vec{\sigma}) = \int_o^\tau d\tau_1\, \Lambda^\tau{}_\tau(\tau_1,\vec{\sigma}) - \Lambda^\tau{}_\tau(\tau,\vec{\sigma})\, \tau,$$

$$V^r(\tau,\vec{\sigma}) = \int_o^\tau d\tau_1\, \Lambda^r{}_\tau(\tau_1,\vec{\sigma}) - \Lambda^r{}_\tau(\tau,\vec{\sigma})\, \tau. \tag{3.7}$$

Let us study the conditions imposed by Eqs.(1.1) on the foliations of the type (3.5) (for the others it is similar) to find which ones correspond to allowed 4-coordinate transformations. We shall represent each Lorentz matrix Λ as the product of a Lorentz boost B and a rotation matrix $\mathcal{R}$ *to separate the translational from the rotational effects* ($\vec{\beta} = \vec{v}/c$ are the boost parameters, $\gamma(\vec{\beta}) = 1/\sqrt{1-\vec{\beta}^2}$, $\vec{\beta}^2 = (\gamma^2-1)/\gamma^2$, $B^{-1}(\vec{\beta}) = B(-\vec{\beta})$; α, β, γ are three Euler angles and $R^{-1} = R^T$)

---

[33] $g_{oo} = \boldsymbol{\epsilon}([(1+\frac{\vec{a}\cdot\vec{x}}{c^2})^2 - \frac{(\omega\times\vec{x})^2}{c^2})$, $g_{oi} = -\boldsymbol{\epsilon}\, \frac{1}{c}\, (\vec{\omega}\times\vec{x})^i$, $g_{ij} = -\boldsymbol{\epsilon}\, \delta_{ij}$, where $\vec{a}$ is the time-dependent acceleration of the observer's frame of reference relative to the comoving inertial frame and $\vec{\omega}$ is the time-dependent angular velocity of the observer's spatial rotation with respect to the comoving frame; $\vec{x}$ is the position vector of a spatial point with respect to the origin of the observer's accelerated frame.

$$\Lambda(\tau,\vec{\sigma}) = B(\vec{\beta}(\tau,\vec{\sigma}))\,\mathcal{R}(\alpha(\tau,\vec{\sigma}),\beta(\tau,\vec{\sigma}),\gamma(\tau,\vec{\sigma})),$$

$$B^A{}_B(\vec{\beta}) = \begin{pmatrix} \gamma(\vec{\beta}) & \gamma(\vec{\beta})\,\beta^s \\ \gamma(\vec{\beta})\,\beta^r & \delta^{rs} + \frac{\gamma^2(\vec{\beta})\,\beta^r\,\beta^s}{\gamma(\vec{\beta})+1} \end{pmatrix}, \qquad \mathcal{R}^A{}_B(\alpha,\beta,\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & R^r{}_s(\alpha,\beta,\gamma) \end{pmatrix},$$

$$R(\alpha,\beta,\gamma) = \tag{3.8}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\beta\cos\gamma \\ -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\beta\sin\gamma \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix}.$$

Then we get

$$\begin{aligned} z^\mu_\tau(\tau,\vec{\sigma}) &= U^\mu_\tau(\tau,\vec{\sigma}) + \partial_\tau\, U^\mu_A(\tau,\vec{\sigma})\,\sigma^A = \\ &= U^\mu_\tau(\tau,\vec{\sigma}) + U^\mu_B(\tau,\vec{\sigma})\,\Omega^B{}_A(\tau,\vec{\sigma})\,\sigma^A, \\ z^\mu_r(\tau,\vec{\sigma}) &= U^\mu_r(\tau,\vec{\sigma}) + \partial_r\, U^\mu_A(\tau,\vec{\sigma})\,\sigma^A = \\ &= U^\mu_r(\tau,\vec{\sigma}) + U^\mu_B(\tau,\vec{\sigma})\,\Omega^B_{(r)A}(\tau,\vec{\sigma})\,\sigma^A, \\ l^\mu(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{|\det\, g_{rs}(\tau,\vec{\sigma})|}}\,\epsilon^\mu{}_{\alpha\beta\gamma}\,[z^\alpha_1\, z^\beta_2\, z^\gamma_3](\tau,\vec{\sigma}), \\ &\quad \text{(normal to the simultaneity surfaces)}, \end{aligned} \tag{3.9}$$

where we have introduced the following matrices

$$\Omega^A{}_B = (\Lambda^{-1}\,\partial_\tau\,\Lambda)^A{}_B = (\mathcal{R}^{-1}\,\partial_\tau\,\mathcal{R} + \mathcal{R}^{-1}\,B^{-1}\,\partial_\tau\,B\,\mathcal{R})^A{}_B = (\Omega_{\mathcal{R}} + \mathcal{R}^{-1}\,\Omega_B\,\mathcal{R})^A{}_B,$$

$$\Omega_{\mathcal{R}} = \mathcal{R}^{-1}\,\partial_\tau\,\mathcal{R} = \begin{pmatrix} 0 & 0 \\ 0 & \Omega_R = R^{-1}\,\partial_\tau\,R \end{pmatrix},$$

$$\Omega_B = B^{-1}(\vec{\beta})\, \partial_\tau\, B(\vec{\beta}) = -\partial_\tau\, B(-\vec{\beta})\, B^{-1}(-\vec{\beta}) =$$

$$= \begin{pmatrix} 0 & \gamma\, (\partial_\tau \beta^s + \frac{\gamma^2\, \vec{\beta}\cdot \partial_\tau \vec{\beta}\, \beta^s}{\gamma+1}) \\ \gamma\, (\partial_\tau \beta^r + \frac{\gamma^2\, \vec{\beta}\cdot \partial_\tau \vec{\beta}\, \beta^r}{\gamma+1}) & -\frac{\gamma^2}{\gamma+1}\, (\beta^r\, \partial_\tau \beta^s - \partial_\tau \beta^r\, \beta^s) \end{pmatrix},$$

$$\Omega = \begin{pmatrix} 0 & \gamma\, (\partial_\tau \beta^u + \frac{\gamma^2\, \vec{\beta}\cdot \partial_\tau \vec{\beta}\, \beta^u}{\gamma+1})\, R^u{}_s \\ R^{Tr}{}_u\, \gamma\, (\partial_\tau \beta^u + \frac{\gamma^2\, \vec{\beta}\cdot \partial_\tau \vec{\beta}\, \beta^u}{\gamma+1}) & \Omega^r_{R\, s} - \frac{\gamma^2}{\gamma+1}\, R^{T\, r}{}_u\, (\beta^u\, \partial_\tau \beta^v - \partial_\tau \beta^u\, \beta^v)\, R^v{}_s \end{pmatrix},$$

$$\Omega^A_{(r)B} = (\Lambda^{-1}\, \partial_r\, \Lambda)^A{}_B = (\mathcal{R}^{-1}\, \partial_r\, \mathcal{R} + \mathcal{R}^{-1}\, B^{-1}\, \partial_r\, B\, \mathcal{R})^A{}_B = (\Omega_{\mathcal{R}\,(r)} + \mathcal{R}^{-1}\, \Omega_{B\,(r)}\, \mathcal{R})^A{}_B =$$

$$= \begin{pmatrix} 0 & \gamma\, (\partial_r \beta^u + \frac{\gamma^2\, \vec{\beta}\cdot \partial_r \vec{\beta}\, \beta^u}{\gamma+1})\, R^u{}_s \\ R^{Tw}{}_u\, \gamma\, (\partial_r \beta^u + \frac{\gamma^2\, \vec{\beta}\cdot \partial_r \vec{\beta}\, \beta^u}{\gamma+1}) & \Omega^w_{R\, s} - \frac{\gamma^2}{\gamma+1}\, R^{T\, w}{}_u\, (\beta^u\, \partial_r \beta^v - \partial_r \beta^u\, \beta^v)\, R^v{}_s \end{pmatrix}, \tag{3.10}$$

assumed to vanish at spatial infinity, $\Omega^A{}_B(\tau, \vec{\sigma}), \Omega^A_{(r)\, B}(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} 0$. *The matrix Ω_B describes the translational velocity ($\vec{\beta}$) and acceleration ($\partial_\tau \vec{\beta}$), while the matrix Ω_R the rotational angular velocity.*

The z^μ_A's and the associated 4-metric are

$$\begin{aligned} z^\mu_\tau(\tau, \vec{\sigma}) \;&=\; \Big([1 + \Omega^\tau{}_r\, \sigma^r]\, U^\mu_\tau + \Omega^r{}_A\, \sigma^A\, U^\mu_r \Big)(\tau, \vec{\sigma}), \\ z^\mu_r(\tau, \vec{\sigma}) \;&=\; \Big(\Omega^\tau_{(r)\, s}\, \sigma^s\, U^\mu_\tau + [\delta^s_r + \Omega^s_{(r)\, A}\, \sigma^A]\, U^\mu_s \Big)(\tau, \vec{\sigma}), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} g_{\tau\tau}(\tau, \vec{\sigma}) \;&=\; \Big(z^\mu_\tau\, \eta_{\mu\nu}\, z^\nu_\tau \Big)(\tau, \vec{\sigma}) = \boldsymbol{\epsilon} \Big([1 + \Omega^\tau{}_r\, \sigma^r]^2 - \sum_r [\Omega^r{}_A\, \sigma^A]^2 \Big)(\tau, \vec{\sigma}), \\ g_{r\tau}(\tau, \vec{\sigma}) \;&=\; \Big(z^\mu_r\, \eta_{\mu\nu}\, z^\nu_\tau \Big)(\tau, \vec{\sigma}) = \boldsymbol{\epsilon} \Big(\Omega^\tau_{(r)\, s}\, \sigma^s\, [1 + \Omega^\tau{}_u\, \sigma^u] - \\ &-\; \sum_s \Omega^s{}_A\, \sigma^A\, [\delta^s_r + \Omega^s_{(r)\, A}\, \sigma^A] \Big)(\tau, \vec{\sigma}), \end{aligned}$$

$$
\begin{aligned}
g_{rs}(\tau,\vec{\sigma}) &= \Big(z^\mu_r\,\eta_{\mu\nu}\,z^\nu_s\Big)(\tau,\vec{\sigma}) = \boldsymbol{\epsilon}\Big(-\delta_{rs} - [\Omega^r_{(s)\,A} + \Omega^s_{(r)\,A}]\,\sigma^A + \\
&+\ \Omega^\tau_{(r)\,u}\,\Omega^\tau_{(s)\,v}\,\sigma^u\,\sigma^v - \sum_u \Omega^u_{(r)\,A}\,\Omega^u_{(s)\,A}\,\sigma^A\,\sigma^B\Big)(\tau,\vec{\sigma}).
\end{aligned}
\tag{3.12}
$$

Eqs.(1.1) are complicated restrictions on the parameters $\vec{\beta}(\tau,\vec{\sigma})$, $\alpha(\tau,\vec{\sigma})$, $\beta(\tau,\vec{\sigma})$, $\gamma(\tau,\vec{\sigma})$ of the Lorentz transformations, which say that *translational accelerations and rotational frequencies are not independent* but must *balance each other* if Eqs.(3.5) describe the inverse of an allowed 4-coordinate transformation.

Let us consider two extreme cases.

A) *Rigid non-inertial reference frames with translational acceleration exist.* An example are the following embeddings, which are compatible with the locality hypothesis only for $f(\tau) = \tau$ (this corresponds to $\Lambda = B(\vec{0})\,\mathcal{R}(0,0,0)$, i.e. to an inertial reference frame)

$$
\begin{aligned}
z^\mu(\tau,\vec{\sigma}) &= x^\mu_o + \epsilon^\mu_\tau\,f(\tau) + \epsilon^\mu_r\,\sigma^r,\\
g_{\tau\tau}(\tau,\vec{\sigma}) = \boldsymbol{\epsilon}\Big(\frac{df(\tau)}{d\tau}\Big)^2, &\quad g_{\tau r}(\tau,\vec{\sigma}) = 0,\quad g_{rs}(\tau,\vec{\sigma}) = -\boldsymbol{\epsilon}\,\delta_{rs}.
\end{aligned}
\tag{3.13}
$$

This is a foliation with parallel hyper-planes with respect to a centroid $x^\mu(\tau) = x^\mu_o + \epsilon^\mu_\tau\,f(\tau)$ (origin of 3-coordinates). The hyper-planes have translational acceleration $\ddot{x}^\mu(\tau) = \epsilon^\mu_\tau\,\ddot{f}(\tau)$, so that they are not uniformly distributed like in the inertial case $f(\tau) = \tau$.

B) On the other hand *rigid rotating reference frames do not exist.* Let us consider the embedding (compatible with the locality hypothesis) with $\Lambda = B(\vec{0})\,\mathcal{R}(\alpha(\tau),\beta(\tau),\gamma(\tau))$ and $x^\mu(\tau) = x^\mu_o + \epsilon^\mu_\tau\,\tau$

$$
\begin{aligned}
z^\mu(\tau,\vec{\sigma}) &= x^\mu(\tau) + \epsilon^\mu_r\,R^r{}_s(\tau)\,\sigma^s,\\
z^\mu_\tau(\tau,\vec{\sigma}) &= \dot{x}^\mu(\tau) + \epsilon^\mu_r\,\dot{R}^r{}_s(\tau)\,\sigma^s, \qquad z^\mu_r(\tau) = \epsilon^\mu_s\,R^s{}_r(\tau),\\
g_{\tau\tau}(\tau,\vec{\sigma}) &= \boldsymbol{\epsilon}\Big(\dot{x}^2(\tau) + 2\,\dot{x}_\mu(\tau)\,\epsilon^\mu_r\,\dot{R}^r{}_s(\tau)\,\sigma^s - \boldsymbol{\epsilon}\,\dot{R}^r{}_u(\tau)\,\dot{R}^r{}_v(\tau)\,\sigma^u\,\sigma^v\Big),
\end{aligned}
$$

$$\begin{aligned} g_{\tau r}(\tau, \vec{\sigma}) &= \boldsymbol{\epsilon} \Big(\dot{x}_\mu(\tau)\, \epsilon^\mu_s\, R^s{}_r(\tau) - \epsilon\, \dot{R}^v{}_u(\tau)\, R^v{}_r(\tau)\, \sigma^u \Big), \\ g_{rs}(\tau, \vec{\sigma}) &= -\boldsymbol{\epsilon}\, R^u{}_r(\tau)\, R^u{}_s(\tau), \end{aligned} \tag{3.14}$$

which corresponds to a foliation with parallel space-like hyper-planes with normal $l^\mu = \epsilon^\mu_\tau$. It can be verified that it is not the inverse of an allowed 4-coordinate transformation, because the associated $g_{\tau\tau}(\tau, \vec{\sigma})$ has a zero at [34]

$$\sigma = \sigma_R = \frac{1}{\Omega(\tau)} \Big[- \dot{x}_\mu(\tau)\, b^\mu_r(\tau)\, (\hat{\sigma} \times \hat{\Omega}(\tau))^r + \sqrt{\dot{x}^2(\tau) + [\dot{x}_\mu(\tau)\, b^\mu_r(\tau)\, (\hat{\sigma} \times \hat{\Omega}(\tau))^2]^2}\, \Big], \tag{3.15}$$

with $\sigma_R \rightarrow \infty$ for $\Omega \rightarrow 0$. At $\sigma = \sigma_R$ the time-like vector $z^\mu_\tau(\tau, \vec{\sigma})$ becomes light-like (the *horizon problem*), while for an admissible foliation with space-like leaves it must always remain time-like.

This pathology (the so-called horizon problem) is common to most of the rotating coordinate systems quoted in Subsection D of the Introduction. Let us remark that an analogous pathology happens on the event horizon of the Schwarzschild black hole. Also in this case we have a coordinate singularity where the time-like Killing vector of the static space-time becomes light-like. For the rotating Kerr black hole the same coordinate singularity happens already at the boundary of the ergosphere [96]. Also in the existing theory of rotating relativistic stars [97], where differential rotations are replacing the rigid ones in model building, it is assumed that in certain rotation regimes an ergosphere may form [98]: again, if one uses 4-coordinates adapted to the Killing vectors, one gets a similar coordinate singularity.

In the next Section we shall consider the minimal modification of Eq.(3.14) so to obtain the inverse of an allowed 4-coordinate transformation.

[34] We use the notations $\vec{\sigma} = \sigma\, \hat{\sigma}$, $\sigma = |\vec{\sigma}|$, $\vec{\Omega} = \Omega\, \hat{\Omega}$, $\hat{\sigma}^2 = \hat{\Omega}^2 = 1$, $\Omega^u = -\frac{1}{2}\, \epsilon^{urs}\, (\dot{R}\, R^{-1})^r{}_s$, $b^\mu_r(\tau) = \epsilon^\mu_s\, R_r{}^s(\tau)$.

IV. THE SIMPLEST NOTION OF SIMULTANEITY WHEN ROTATIONS ARE PRESENT.

Let us look for the simplest embedding $x^\mu = z^\mu(\tau, \vec{\sigma})$, inverse of an admissible 4-coordinate transformation $x^\mu \mapsto \sigma^A$ compatible with the locality hypothesis, which contains a rotating reference frame, with also translational acceleration, of the type of Eq.(3.12). The minimal modification of Eq.(3.12) is to replace the rotation matrix $R(\tau)$ with $R(\tau, |\vec{\sigma}|)$, namely the rotation varies as a function of some radial distance $|\vec{\sigma}|$ (*differential rotation*) from the arbitrary time-like world-line $x^\mu(\tau)$, origin of the 3-coordinates on the simultaneity surfaces. Since the 3-coordinates σ^r are Lorentz scalar we shall use the radial distance $\sigma = |\vec{\sigma}| = \sqrt{\delta_{rs}\,\sigma^r\,\sigma^s}$, so that $\sigma^r = \sigma\,\hat{\sigma}^r$ with $\delta_{rs}\,\hat{\sigma}^r\,\hat{\sigma}^s = 1$. Therefore let us replace Eq.(3.12) with the following embedding

$$
\begin{aligned}
z^\mu(\tau, \vec{\sigma}) &= x^\mu(\tau) + \epsilon^\mu_r\, R^r{}_s(\tau, \sigma)\, \sigma^s \stackrel{def}{=} x^\mu(\tau) + b^\mu_r(\tau, \sigma)\, \sigma^r,\\
R^r{}_s(\tau, \sigma) &\rightarrow_{\sigma\rightarrow\infty} \delta^r_s, \qquad \partial_A\, R^r{}_s(\tau, \sigma) \rightarrow_{\sigma\rightarrow\infty} 0,\\
b^\mu_s(\tau, \sigma) &= \epsilon^\mu_r\, R^r{}_s(\tau, \sigma) \rightarrow_{\sigma\rightarrow\infty} \epsilon^\mu_s, \quad [b^\mu_r\, \eta_{\mu\nu}\, b^\nu_s](\tau, \sigma) = -\boldsymbol{\epsilon}\, \delta_{rs}.
\end{aligned} \tag{4.1}
$$

Since $z^\mu_r(\tau, \vec{\sigma}) = \epsilon^\mu_s\, \partial_r\, [R^s{}_u(\tau, \sigma)\, \sigma^u]$, it follows that the normal to the simultaneity surfaces is $l^\mu = \epsilon^\mu_\tau$, namely the hyper-surfaces are *parallel space-like hyper-planes*. These hyper-planes have translational acceleration $\ddot{x}^\mu(\tau)$ and a rotating 3-coordinate system with rotational frequency

$$
\begin{aligned}
\Omega^r(\tau, \sigma) \;&=\; -\frac{1}{2}\, \epsilon^{ruv}\, \Big[R^{-1}(\tau, \sigma)\, \frac{\partial R(\tau, \sigma)}{\partial \tau}\Big]^{uv} \rightarrow_{\sigma\rightarrow\infty} 0,\\
&\Downarrow\\
\frac{\partial b^\mu_s(\tau, \sigma)}{\partial \tau} \;&=\; \epsilon^\mu_r\, \frac{\partial R^r{}_s(\tau, \sigma)}{\partial \tau} = -\epsilon^{suv}\, \Omega^u(\tau, \sigma)\, b^\mu_v(\tau, \sigma),
\end{aligned}
$$

$$\Omega^1(\tau,\sigma) = \Big[\partial_\tau\beta\, \sin\, \gamma - \partial_\tau\alpha\, \sin\, \beta\, \cos\, \gamma\Big](\tau,\sigma),$$

$$\Omega^2(\tau,\sigma) = \Big[\partial_\tau\beta\, \cos\, \gamma + \partial_\tau\alpha\, \sin\, \beta\, \sin\, \gamma\Big](\tau,\sigma),$$

$$\Omega^3(\tau,\sigma) = \Big[\partial_\tau\gamma + \partial_\tau\alpha\, \cos\, \beta\Big](\tau,\sigma). \qquad (4.2)$$

In the last three lines we used Eqs.(3.8) to find the angular velocities. Moreover we can define

$$\Omega_{(r)}(\tau,\sigma) = \Big[R^{-1}\,\partial_r\, R\Big](\tau,\sigma) = 2\hat{\sigma}^r\,\Big[R^{-1}\,\frac{\partial R}{\partial\sigma}\Big](\tau,\sigma) \rightarrow_{\sigma\rightarrow\infty} 0,$$

$$\Omega^u_{(r)v}(\tau,\sigma)\,\sigma^v = \Phi_{uv}(\tau,\sigma)\,\frac{\sigma^r\,\sigma^v}{\sigma}, \qquad \Phi_{uv} = -\Phi_{vu}, \qquad (4.3)$$

As a consequence we have

$$\dot{x}^\mu(\tau) = \boldsymbol{\epsilon}\,\Big([\dot{x}_\nu(\tau)\,l^\nu]\,l^\mu - \sum_r\,[\dot{x}_\nu(\tau)\,\epsilon^\nu_r]\,\epsilon^\mu_r\Big),$$

$$z^\mu_\tau(\tau,\vec{\sigma}) = N(\tau,\vec{\sigma})\,l^\mu + N^r(\tau,\vec{\sigma})\,z^\mu_r(\tau,\vec{\sigma}) =$$

$$= \dot{x}^\mu(\tau) - \epsilon^{suv}\,\Omega^u(\tau,\sigma)\,b^\mu_v(\tau,\sigma)\,\sigma^s =$$

$$= \dot{x}^\mu(\tau) - (\vec{\sigma}\times\vec{\Omega}(\tau,\sigma))^r\,b^\mu_r(\tau,\sigma) \rightarrow_{\sigma\rightarrow\infty} \dot{x}^\mu(\tau),$$

$$z^\mu_r(\tau,\vec{\sigma}) = \epsilon^\mu_s\,\Big[R^s{}_r(\tau,\sigma) + \partial_r\,R^s{}_u(\tau,\sigma)\,\sigma^u\Big] =$$

$$= b^\mu_s(\tau,\sigma)\,\Big[\delta^s_r + \Omega^s_{(r)u}(\tau,\sigma)\,\sigma^u\Big] \rightarrow_{\sigma\rightarrow\infty} \epsilon^\mu_r, \qquad (4.4)$$

and then we obtain

$$
\begin{aligned}
g_{\tau\tau}(\tau,\vec{\sigma}) &= \dot{x}^2(\tau) - 2\,\dot{x}_\mu(\tau)\, b^\mu_r(\tau,\sigma)\,(\vec{\sigma}\times\vec{\Omega}(\tau,\sigma))^r - \boldsymbol{\epsilon}\,(\vec{\sigma}\times\vec{\Omega})^2 = \\
&= \Big[N^2 - g_{rs}\, N^r\, N^s\Big](\tau,\vec{\sigma}),\\
g_{\tau r}(\tau,\vec{\sigma}) &= \Big[g_{rs}\, N^s\Big](\tau,\vec{\sigma}) = \\
&= \dot{x}_\mu(\tau)\, b^\mu_r(\tau,\sigma)\,\Big[\delta^v_r + \Omega^v_{(r)u}(\tau,\sigma)\,\sigma^u\Big] + \boldsymbol{\epsilon}\,[\vec{\sigma}\times\vec{\Omega}(\tau,\sigma)]^s\,\Big[\delta^s_r + \Omega^s_{(r)u}(\tau,\sigma)\,\sigma^u\Big],\\
-\boldsymbol{\epsilon}\, g_{rs}(\tau,\vec{\sigma}) &= \delta_{rs} + \Big(\Omega^r_{(s)u}(\tau,\sigma) + \Omega^s_{(r)u}(\tau,\sigma)\Big)\,\sigma^u + \sum_w \Omega^w_{(r)u}(\tau,\sigma)\,\Omega^w_{(s)v}(\tau,\sigma)\,\sigma^u\,\sigma^v.
\end{aligned}
\tag{4.5}
$$

The requirement that $g_{\tau\tau}(\tau,\vec{\sigma})$ and $g_{\tau r}(\tau,\vec{\sigma})$ tend to finite limits at spatial infinity puts the restrictions

$$
|\vec{\Omega}(\tau,\sigma)|, \qquad |\Omega^u_{(r)v}(\tau,\sigma)| \rightarrow_{\sigma\rightarrow\infty} O(\sigma^{-(1+\eta)}), \quad \eta > 0,
$$

$$
\Downarrow
$$

$$
\begin{aligned}
\partial_A\, R^r{}_s(\tau,\sigma) &\rightarrow_{\sigma\rightarrow\infty} O(\sigma^{-(1+\eta)}), \Rightarrow\; R^r{}_s(\tau,\sigma) \rightarrow_{\sigma\rightarrow\infty} O(\sigma^{-(1+\eta)}),\\
z^\mu_\tau(\tau,\vec{\sigma}) &\rightarrow_{\sigma\rightarrow\infty} \dot{x}^\mu(\tau) + O(\sigma^{-\eta}),\\
b^\mu_r(\tau,\sigma) &\rightarrow_{\sigma\rightarrow\infty} \epsilon^\mu_r + O(\sigma^{-(1+\eta)}), \qquad z^\mu_r(\tau,\vec{\sigma}) \rightarrow_{\sigma\rightarrow\infty} \epsilon^\mu_r + O(\sigma^{-(1+\eta)}),\\
N^r(\tau,\vec{\sigma})\, z^\mu_r(\tau,\vec{\sigma}) &\rightarrow_{\sigma\rightarrow\infty} -\boldsymbol{\epsilon}\,\dot{x}_\nu(\tau)\,\epsilon^\nu_r\,\epsilon^\mu_r + O(\sigma^{-(1+2\eta)}),\\
N^r(\tau,\vec{\sigma}) &\rightarrow_{\sigma\rightarrow\infty} -\boldsymbol{\epsilon}\,\delta^{rs}\,\dot{x}_\nu(\tau)\,\epsilon^\nu_s + O(\sigma^{-\eta}),
\end{aligned}
$$

$$
\begin{aligned}
N(\tau,\vec{\sigma}) \qquad & l^\mu = [z^\mu_\tau - N^r\, z^\mu_r](\tau,\vec{\sigma}) \rightarrow_{\sigma\rightarrow\infty} \boldsymbol{\epsilon}\, [\dot{x}_\nu(\tau)\, l^\nu]\, l^\mu + O(\sigma^{-\eta}),\\
g_{\tau\tau}(\tau,\vec{\sigma}) \rightarrow_{\sigma\rightarrow\infty}\ & \dot{x}^2(\tau) + O(\sigma^{-2\eta}),\\
g_{\tau r}(\tau,\vec{\sigma}) \rightarrow_{\sigma\rightarrow\infty}\ & \dot{x}_\mu(\tau)\, \epsilon^\mu_r + O(\sigma^{-\eta}),\\
g_{rs}(\tau,\vec{\sigma}) \rightarrow_{\sigma\rightarrow\infty}\ & -\boldsymbol{\epsilon}\, \delta_{rs} + O(\sigma^{-\eta}).
\end{aligned}
\tag{4.6}
$$

Let us look for a family of rotation matrices $R^r{}_s(\tau,\sigma)$ satisfying the condition $\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau,\vec{\sigma}) > 0$ of Eqs.(1.1).

Let us make the ansatz that the Euler angles of $R(\alpha,\beta,\gamma)$ have the following factorized dependence on τ and σ

$$
\alpha(\tau,\sigma) = F(\sigma)\, \tilde{\alpha}(\tau), \qquad \beta(\tau,\sigma) = F(\sigma)\, \tilde{\beta}(\tau), \qquad \gamma(\tau,\sigma) = F(\sigma)\, \tilde{\gamma}(\tau), \tag{4.7}
$$

with

$$
F(\sigma) > 0, \quad \frac{dF(\sigma)}{d\sigma} \neq 0, \qquad F(\sigma) \rightarrow_{\sigma\rightarrow\infty} O(\sigma^{-(1+\eta)}). \tag{4.8}
$$

We get

$$\Omega^1(\tau,\sigma) = F(\sigma)\Big(\dot{\tilde{\beta}}(\tau)\,\sin[F(\sigma)\,\tilde{\gamma}(\tau)] - \dot{\tilde{\alpha}}(\tau)\,\sin[F(\sigma)\,\tilde{\beta}(\tau)]\,\cos[F(\sigma)\,\tilde{\gamma}(\tau)]\Big),$$

$$\Omega^2(\tau,\sigma) = F(\sigma)\Big(\dot{\tilde{\beta}}(\tau)\,\cos[F(\sigma)\,\tilde{\gamma}(\tau)] + \dot{\tilde{\alpha}}(\tau)\,\sin[F(\sigma)\,\tilde{\beta}(\tau)]\,\sin[F(\sigma)\,\tilde{\gamma}(\tau)]\Big),$$

$$\Omega^3(\tau,\sigma) = F(\sigma)\Big(\dot{\tilde{\gamma}}(\tau) + \dot{\tilde{\alpha}}(\tau)\,\cos[F(\sigma)\,\tilde{\beta}(\tau)]\Big),$$

$$\Downarrow$$

$$\Omega^r(\tau,\sigma) = F(\sigma)\,\tilde{\Omega}(\tau,\sigma)\,\hat{n}^r(\tau,\sigma), \qquad \hat{n}^2(\tau,\sigma)=1,$$

$$0 < \tilde{\Omega}(\tau,\sigma) \leq 2\,\max\Big(\dot{\tilde{\alpha}}(\tau),\dot{\tilde{\beta}}(\tau),\dot{\tilde{\gamma}}(\tau)\Big) = 2\,M_1. \tag{4.9}$$

Since $l^\mu = \epsilon^\mu_\tau \stackrel{def}{=} b^\mu_\tau$ and $b^\mu_r(\tau,\sigma)$ form an orthonormal tetrad $[b^\mu_A(\tau,\sigma)\,\eta_{\mu\nu}\,b^\nu_B(\tau,\sigma) = \eta_{AB}]$, let us decompose the future time-like 4-velocity $\dot{x}^\mu(\tau)$ on it ($v_l(\tau)$ is the asymptotic lapse function)

$$\dot{x}^\mu(\tau) = v_l(\tau)\,l^\mu - \sum_r v_r(\tau,\sigma)\,b^\mu_r(\tau,\sigma)$$

$$v_l(\tau) = \boldsymbol{\epsilon}\,\dot{x}_\mu(\tau)\,l^\mu > 0, \qquad v_r(\tau,\sigma) = \boldsymbol{\epsilon}\,\dot{x}_\mu(\tau)\,b^\mu_r(\tau,\sigma),$$

$$\boldsymbol{\epsilon}\,\dot{x}^2(\tau) = v_l^2(\tau) - \sum_r v_r^2(\tau,\sigma) > 0, \Rightarrow \sum_r v_r^2(\tau,\sigma) = \vec{v}^2(\tau,\sigma) \equiv \vec{v}^2(\tau) < v_l^2(\tau), \tag{4.10}$$

We add the condition

$$|\vec{v}(\tau)| \leq \frac{v_l(\tau)}{K}, \qquad K > 1. \tag{4.11}$$

This condition is slightly stronger than the last of Eqs.(4.10), which does not exclude the possibility that the observer in $\vec{\sigma}=0$ has a time-like 4-velocity $\dot{x}^\mu(\tau)$ which, however, becomes light-like at $\tau=\pm\infty$ [35]. The condition (4.11) excludes this possibility. In other

[35] This is the case of a (non-time-like) Rindler observer with uniform 4-acceleration, see Ref.[99]

words the condition (4.11) tell us that the observer is without event-horizon, namely he can explore all the Minkowski space-time by light-signal.

Then the condition $\epsilon\, g_{\tau\tau}(\tau, \vec{\sigma}) > 0$ becomes

$$
\begin{aligned}
&\epsilon\, g_{\tau\tau}(\tau, \vec{\sigma}) = \\
&= \epsilon\, \dot{x}^2(\tau) - 2\, \sigma\, F(\sigma)\, \tilde{\Omega}(\tau, \sigma) \sum_r v_r(\tau, \sigma) \Big[\hat{\sigma} \times \hat{n}(\tau, \sigma)\Big]^r - \sigma^2\, \tilde{\Omega}^2(\tau, \sigma)\, F^2(\sigma) \Big[\hat{\sigma} \times \hat{n}(\tau, \sigma)\Big]^2 = \\
&= c^2(\tau) - 2\, b(\tau, \vec{\sigma})\, X(\tau, \sigma) - a^2(\tau, \vec{\sigma})\, X^2(\tau, \sigma) > 0
\end{aligned}
\tag{4.12}
$$

where we have defined

$$
\begin{aligned}
&c^2(\tau) = \epsilon\, \dot{x}^2(\tau) = v_l^2(\tau) - \vec{v}^2(\tau) > 0, \qquad c^2(\tau) \geq \frac{K^2 - 1}{K^2}\, v_l^2(\tau),\\
&b(\tau, \vec{\sigma}) = \sum_r v_r(\tau, \sigma) \Big[\hat{\sigma} \times \hat{n}(\tau, \sigma)\Big]^r,\\
&|b(\tau, \vec{\sigma})| \leq |\vec{v}(\tau)| < v_l(\tau), \;\; \text{or} \;\; |b(\tau, \vec{\sigma})| \leq \frac{v_l(\tau)}{K}, \;\; K > 1,\\
&a^2(\tau, \vec{\sigma}) = \Big[\hat{\sigma} \times \hat{n}(\tau, \sigma)\Big]^2 > 0, \quad a^2(\tau, \vec{\sigma}) \leq 1, \quad b^2(\tau, \vec{\sigma}) + a^2(\tau, \sigma)\, c^2(\tau) > 0,\\
&X(\tau, \sigma) = \sigma\, F(\sigma)\, \tilde{\Omega}(\tau, \sigma).
\end{aligned}
\tag{4.13}
$$

The study of the equation $a^2\, X^2 + 2\, b\, X - c^2 = A^2\, (X - X_+)\, (X - X_-) = 0$, with solutions $X_\pm = \frac{1}{a^2}\, (-b \pm \sqrt{b^2 + a^2\, c^2})$, shows that $\epsilon\, g_{\tau\tau} > 0$ implies $X_- < X < X_+$; being $X_- < 0$ and $X > 0$ [see Eq.(4.9)], we have that a half of the conditions ($X_- < X$) is always satisfied. We have only to discuss the condition $X < X_+$.

Since $-v_l/K \leq b \leq v_l/K$, when b increases in this interval X_+ decrease with b. This implies

$$
X_+ > \frac{1}{a^2} \Big(- \frac{v_l}{K} + \sqrt{\frac{v_l^2}{K^2} + a^2\, c^2} \Big),
$$

so that $c^2 \geq \frac{K^2 - 1}{K^2}\, v_l^2$ implies that we will have $g_{\tau\tau} > 0$ if $0 < X < \frac{v_l}{K\, a^2}\, (\sqrt{1 + (K^2 - 1)\, a^2} - 1)$, namely if the function $F(\sigma)$ satisfies the condition

$$|F(\sigma)| < \frac{v_l(\tau)}{K\,\sigma\,a^2(\tau,\vec{\sigma})\,\tilde{\Omega}(\tau,\sigma)}\Big(\sqrt{1+(K^2-1)\,a^2(\tau,\vec{\sigma})}-1\Big) = \frac{v_l(\tau)}{K\,\tilde{\Omega}(\tau,\sigma)}\,g(a^2).$$

Since $a^2 \leq 1$ and $g(x) = (1/x)(\sqrt{1+(K^2-1)x}-1)$ is decreasing for x increasing in the interval $0 < x < 1$ ($K > 1$), we get $g(a^2) > g(1) = K-1$ and the stronger condition

$$|F(\sigma)| < \frac{v_l(\tau)}{K\,\tilde{\Omega}(\tau,\sigma)}\,(K-1).$$

The condition (4.9) on the Euler angles and the fact that Eq.(4.11) implies $\min v_l(\tau) = m > 0$ lead to the following final condition on $F(\sigma)$

$$\begin{aligned}
& 0 < F(\sigma) < \frac{m}{2\,K\,M_1\,\sigma}\,(K-1) = \frac{1}{M\,\sigma}, \qquad \frac{dF(\sigma)}{d\sigma} \neq 0,\\
\text{or}\qquad & |\partial_\tau\alpha(\tau,\sigma)|, |\partial_\tau\beta(\tau,\sigma)|, |\partial_\tau\gamma(\tau,\sigma)| < \frac{m}{2\,K\,\sigma}\,(K-1),\\
\text{or}\qquad & |\Omega^r(\tau,\sigma)| < \frac{m}{K\,\sigma}\,(K-1). \qquad (4.14)
\end{aligned}$$

This means that, while the linear velocities $\dot{x}^\mu(\tau)$ and the translational accelerations $\ddot{x}^\mu(\tau)$ are arbitrary, the allowed rotations $R(\alpha,\beta,\gamma)$ on the leaves of the foliation *have the rotational frequencies*, namely the angular velocities $\Omega^r(\tau,\sigma)$, *limited by an upper bound proportional to the minimum of the linear velocity* $v_l(\tau) = \dot{x}_\mu(\tau)\,l^\mu$ *orthogonal to the parallel hyper-planes.*

Instead of checking the conditions (1.1) on $g_{rs}(\tau,\vec{\sigma})$, let us write

$$\begin{aligned}
z^\mu(\tau,\vec{\sigma}) &= \xi_l(\tau,\vec{\sigma})\,l^\mu - \sum_r \xi_r(\tau,\vec{\sigma})\,\epsilon^\mu_r,\\
\xi_l(\tau,\vec{\sigma}) &= \boldsymbol{\epsilon}\,z_\mu(\tau,\vec{\sigma})\,l^\mu = \boldsymbol{\epsilon}\,x_\mu(\tau)\,l^\mu = x_l(\tau),\\
\xi_r(\tau,\vec{\sigma}) &= \boldsymbol{\epsilon}\,z_\mu(\tau,\vec{\sigma})\,\epsilon^\mu_r = \boldsymbol{\epsilon}\,x_\mu(\tau)\,\epsilon^\mu_r + R^r{}_s(\tau,\sigma)\,\sigma^s = x_{\epsilon\,r}(\tau) + R^r{}_s(\tau,\sigma)\,\sigma^s, \qquad (4.15)
\end{aligned}$$

so that we get

$$\begin{aligned}
\partial_\tau\, \xi_l(\tau,\vec{\sigma}) &= \dot{x}_l(\tau) = v_l(\tau), \qquad \partial_r\, \xi_l(\tau,\vec{\sigma}) = 0,\\
\partial_u\, \xi_r(\tau,\vec{\sigma}) &= R^r{}_u(\tau,\sigma) + \partial_u R^r{}_s(\tau,\sigma)\, \sigma^s =\\
&= R^r{}_v(\tau,\sigma)\, \Big[\delta^v_u + \omega^v_{(u)w}(\tau,\sigma)\, \sigma^w\Big] =\\
&= R^r{}_v(\tau,\sigma)\, \Big[\delta^v_u + \Phi_{uv}(\tau,\sigma)\, \frac{\sigma^u\, \sigma^w}{\sigma}\Big] \stackrel{def}{=} \Big(R(\tau,\sigma)\, M(\tau,\vec{\sigma})\Big)_{ru},
\end{aligned} \tag{4.16}$$

and let us show that $\sigma^A = (\tau,\vec{\sigma}) \mapsto (\xi_l(\tau,\vec{\sigma}), \xi_r(\tau,\vec{\sigma})\,)$ is a coordinate transformation with positive Jacobian. This will ensure that these foliations with parallel hyper-planes are defined by embeddings such that $\sigma^A \mapsto x^\mu = z^\mu(\tau,\vec{\sigma})$ is the inverse of an admissible 4-coordinate transformation $x^\mu \mapsto \sigma^A$.

Therefore we have to study the Jacobian

$$J(\tau,\vec{\sigma}) = \begin{pmatrix} \frac{\partial\, \xi_l(\tau,\vec{\sigma})}{\partial\tau} & \frac{\partial\, \xi_s(\tau,\vec{\sigma})}{\partial\tau}\\ \frac{\partial\, \xi_l(\tau,\vec{\sigma})}{\partial\sigma^r} & \frac{\partial\, \xi_s(\tau,\vec{\sigma})}{\partial\sigma^r} \end{pmatrix} = \begin{pmatrix} v_l(\tau) & \frac{\partial\, \xi_s(\tau,\vec{\sigma})}{\partial\tau}\\ 0_r & \Big(R(\tau,\sigma)\, M(\tau,\vec{\sigma})\Big)_{rs} \end{pmatrix},$$

$$\det\, J(\tau,\vec{\sigma}) = v_l(\tau)\, \det\, R(\tau,\sigma)\, \det\, M(\tau,\vec{\sigma}) = v_l(\tau)\, \det\, M(\tau,\vec{\sigma}). \tag{4.17}$$

To show that $\det\, M(\tau,\vec{\sigma}) \neq 0$, let us look for the null eigenvectors $W_r(\tau,\vec{\sigma})$ of the matrix $M(\tau,\vec{\sigma})$, $M_{rs}(\tau,\vec{\sigma})\, W_s(\tau,\vec{\sigma}) = 0$ or $W_r(\tau,\vec{\sigma}) - \Phi_{uv}(\tau,\sigma)\, \frac{\sigma^u}{\sigma}\, \sigma^s\, W_s(\tau,\vec{\sigma}) = 0$ [see Eq.(4.3)]. Due to $\Phi_{uv} = -\Phi_{vu}$, we get $\sigma^s\, W_s(\tau,\vec{\sigma}) = 0$ and this implies $W_r(\tau,\vec{\sigma}) = 0$, i.e. the absence of null eigenvalues. Therefore $\det\, M(\tau,\vec{\sigma}) \neq 0$ and an explicit calculation shows that $\det\, M(\tau,\vec{\sigma}) = 1$. As a consequence, we get $\det\, J(\tau,\vec{\sigma}) = v_l(\tau) > 0$. Therefore, $x^\mu \mapsto \sigma^A$ is an admissible 4-coordinate transformation.

Since in Eq.(4.1) $x^\mu(\tau)$ is interpretable as the world-line of an arbitrary non-inertial time-like accelerated observer, these allowed foliations with parallel space-like hyper-planes (*not*

orthogonal to the world-line) define good notions of simultaneity, replacing the attempts based on Fermi coordinates, for an accelerated observer with arbitrary time-like world-line $x^\mu(\tau)$.

Let us remark that the congruence of time-like world-lines associated to the constant normal l^μ defines an inertial reference frame: each inertial observer is naturally endowed with the orthonormal tetrad $b^\mu_A = (l^\mu; \epsilon^\mu_r)$.

Let us consider the second skew congruence, whose observer world-lines are $x^\mu_{\vec \sigma}(\tau) = z^\mu(\tau, \vec \sigma)$, and let us look for an orthonormal tetrad $V^\mu_A(\tau, \vec \sigma) = (z^\mu_\tau(\tau, \vec \sigma)/\sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec \sigma)}; V^\mu_r(\tau, \vec \sigma))$ to be associated to each of its time-like observers. Due to the orthonormality we have $V^\mu_A(\tau, \vec \sigma) = \Lambda^\mu{}_{\nu=A}(\tau, \vec \sigma)$ with $\Lambda(\tau, \vec \sigma)$ a Lorentz matrix. Therefore we can identify them with $SO(3,1)$ matrices parametrized as the product of a pure boost with a pure rotation as in Eqs. (III8). If we introduce

$$
\begin{aligned}
&\vec E_r(\tau, \vec \sigma) = \{E^k_r(\tau, \vec \sigma)\} = R^{s=k}_r(\alpha_m(\tau, \sigma), \beta_m(\tau, \sigma), \gamma_m(\tau, \sigma))\\
&\Rightarrow\ \ \frac{\partial \vec E_r(\tau, \vec \sigma)}{\partial \tau} \,{\buildrel {def} \over =}\, \vec \omega_m(\tau) \times \vec E_r(\tau, \vec \sigma),\\
&B^{jk}(\vec \beta_m(\tau, \vec \sigma)) = \delta^{ij} + \frac{\gamma^2(\vec \beta_m(\tau, \sigma))}{\gamma(\vec \beta_m(\tau, \sigma)) + 1}\, \beta^i_m(\tau, \sigma)\, \beta^j_m(\tau, \sigma),
\end{aligned}
\tag{4.18}
$$

we can write

$$
V^\mu_A(\tau, \vec \sigma) = \Lambda^\mu_{\nu=A}(\tau, \vec \sigma) = \begin{pmatrix} \frac{1}{\sqrt{1 - \vec \beta^2_m(\tau, \vec \sigma)}} & \frac{\vec \beta_m(\tau, \vec \sigma) \cdot \vec E_r(\tau, \vec \sigma)}{\sqrt{1 - \vec \beta^2_m(\tau, \vec \sigma)}} \\ \frac{\beta^j_m(\tau, \vec \sigma)}{\sqrt{1 - \vec \beta^2_m(\tau, \vec \sigma)}} & B^{jk}(\vec \beta_m(\tau, \vec \sigma))\, E^k_r(\tau, \vec \sigma) \end{pmatrix}. \tag{4.19}
$$

We stress that for every observer $x^\mu_{\vec \sigma}(\tau)$ the choice of the $V^\mu_r(\tau, \vec \sigma)$'s, and therefore also of the $\vec E_r(\tau, \vec \sigma)$'s, is arbitrary. As a consequence the angular velocity $\vec \omega_m(\tau)$ *defined* by the second of the Eqs.(4.18) is in general not related with the angular velocity (4.2) defined by the embedding. On the contrary, the parameter $\vec \beta_m(\tau, \vec \sigma)$ is related to the embedding by the relation $\beta^i_m(\tau, \vec \sigma) = z^i_\tau(\tau, \vec \sigma)/z^o_\tau(\tau, \vec \sigma)$.

For every observer $x^\mu_{\vec \sigma}(\tau)$ of the congruence, endowed with the orthonormal tetrad $E^\mu_{\vec \sigma\, A}(\tau) = V^\mu_A(\tau, \vec \sigma)$, we get

$$\frac{dE^{\mu}_{\vec\sigma\, A}(\tau)}{d\tau} = {\mathcal{A}_{\vec\sigma\, A}}^{B}(\tau)\, V^{\mu}_{\vec\sigma\, B}(\tau),$$

$$\Rightarrow\ \mathcal{A}_{\vec\sigma\, AB}(\tau) = -\mathcal{A}_{\vec\sigma\, BA}(\tau) = \frac{dE^{\mu}_{\vec\sigma\, A}(\tau)}{d\tau}\, \eta_{\mu\nu} E^{\nu}_{\vec\sigma\, B}(\tau), \tag{4.20}$$

Using the (4.19) we obtain $[\gamma(\tau,\vec\sigma) = 1/\sqrt{1-\vec\beta^2_m(\tau,\vec\sigma)},\ \dot{\vec\beta}_m(\tau,\vec\sigma) = d\vec\beta_m(\tau,\vec\sigma)/d\tau]$

$$\begin{aligned} a_{\vec\sigma\, r}(\tau) &= \mathcal{A}_{\vec\sigma\, \tau r}(\tau) = \Big[-\gamma\, (\dot{\vec\beta}_m \cdot \vec E_r) - \frac{\gamma^3}{\gamma+1}\, (\dot{\vec\beta}_m \cdot \vec\beta_m)(\vec\beta_m \cdot \vec E_r)\Big]\, (\tau,\vec\sigma)\\ \Omega_{\vec\sigma\, r}(\tau) &= \frac{1}{2}\, \epsilon_{ruv}\, \mathcal{A}_{\vec\sigma\, uv}(\tau) =\\ &= \Big[-\vec\omega_m \cdot \vec E_r - \frac{\gamma^2}{\gamma+1} \epsilon^{rsu} (\vec\beta_m \cdot \vec E_s)(\dot{\vec\beta}_m \cdot \vec E_u)\Big]\, (\tau,\vec\sigma) \end{aligned} \tag{4.21}$$

Therefore the acceleration radii (see Subsection C of the Introduction) of these observers are

$$\begin{aligned} I_1 &= \vec\Omega^2_{\vec\sigma} - \vec a^2_{\vec\sigma} = \Big[\vec\omega^2_m + 2\frac{\gamma^2}{\gamma+1} \vec\omega_m \cdot (\dot{\vec\beta}_m \times \vec\beta_m) + \gamma^2(\gamma-2)\, \dot{\vec\beta}^2_m - \frac{\gamma^6}{\gamma+1}\, (\dot{\vec\beta}_m \cdot \vec\beta_m)^2\Big]\, (\tau,\vec\sigma)\\ I_2 &= \vec a_{\vec\sigma} \cdot \vec\Omega_{\vec\sigma} = \Big[\gamma\, (\dot{\vec\beta}_m \cdot \vec\omega_m) + \frac{\gamma^3}{\gamma+1}\, (\dot{\vec\beta}_m \cdot \vec\beta_m)(\vec\beta_m \cdot \vec\omega_m)\Big]\, (\tau,\vec\sigma) \end{aligned} \tag{4.22}$$

The non-relativistic limit of the embedding (4.1) can be obtained by choosing $\epsilon^{\mu}_r = (0; e^i_r)$. We obtain a generalization of the standard translating and rotating 3-coordinate systems on the hyper-planes of constant absolute Newtonian time

$$\begin{aligned} t'(\tau) &= t(\tau),\\ z^i(\tau,\vec\sigma) &= x^i(\tau) + e^i_r\, {R^r}_s(\tau,\sigma)\, \sigma^s, \end{aligned} \tag{4.23}$$

without any restriction on rotations, namely with $R = R(\tau)$ allowed.

V. NOTION OF SIMULTANEITY ASSOCIATED TO ROTATING REFERENCE FRAMES.

In this Section we consider the inverse problem of finding a foliation of Minkowski space-time with simultaneity surfaces associated to a given arbitrary reference frame with non-zero vorticity, namely to a time-like vector field whose expression in Cartesian 4-coordinates in an inertial system is $\tilde{u}^\mu(x)$ with $\tilde{u}^2(x) = \boldsymbol{\epsilon}$. In other words we are looking for embeddings $z^\mu(\tau, \vec{\sigma})$, inverse of an admissible 4-coordinate transformation, such that we have $\tilde{u}^\mu(z(\tau, \vec{\sigma})) = u^\mu(\tau, \vec{\sigma}) = z^\mu_\tau(\tau, \vec{\sigma})/\sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{\sigma})}$. Let us remark that if the vorticity is zero, the vector field $\tilde{u}^\mu(x)$ is surface-forming, there is a foliation whose surfaces have the normal field proportional to $u^\mu(\tau, \vec{\sigma})$ and these surfaces automatically give an admissible foliation with space-like hyper-surfaces of Minkowski space-time.

Let us first show that, given an arbitrary time-like vector field $\tilde{u}^\mu(x)$, the looked for foliation exists. Let us consider the equation

$$\tilde{u}^\mu(x)\, \frac{\partial s(x)}{\partial x^\mu} = 0, \tag{5.1}$$

where $s(x)$ is a scalar field. This equation means that $s(x)$ is constant along the integral lines $x^\mu(s)$ $[dx^\mu(s)/ds = \tilde{u}^\mu(x(s))]$ of the vector field, i.e. it is a comoving quantity, since

$$\frac{ds(x(s))}{ds} = \tilde{u}^\mu(x(s))\, \frac{\partial s}{\partial x^\mu}(x(s)) = 0. \tag{5.2}$$

Since Eq.(5.1) has three independent solutions $s^{(r)}(x)$, $r = 1, 2, 3$, they can be used to identify three coordinates $\sigma^r(x) = s^{(r)}(x)$. Moreover the three 4-vectors $\frac{\partial \sigma^r(x)}{\partial x^\mu}$ are space-like by construction.

Since Minkowski space-time is globally hyperbolic, there exist *time-functions* $\tau(x)$ such that i) $\tau(x) = const.$ defines space-like hyper-surfaces; ii) $\frac{\partial \tau(x)}{\partial x^\mu}$ is a time-like 4-vector.

As a consequence we can build an invertible 4-coordinate transformation $x^\mu \mapsto \sigma^A(x) = (\tau(x), \sigma^r(x))$, with inverse $\sigma^A = (\tau, \sigma^r) \mapsto x^\mu = z^\mu(\tau, \vec{\sigma})$ for every choice of $\tau(x)$. It can be shown that we get always a non-vanishing Jacobian [36]

---

[36] Let us show that the equations

$$\alpha\, \frac{\partial \tau(x)}{\partial x^\mu} + \beta_r\, \frac{\partial \sigma^r(x)}{\partial x^\mu} = 0$$

implies $\alpha = \beta_r = 0$. If we multiply for $\tilde{u}^\mu(x)$, we get $\alpha\, \tilde{u}^\mu(x)\, \frac{\partial \tau(x)}{\partial x^\mu} = 0$. But $\frac{\partial \tau(x)}{\partial x^\mu}$ and $\tilde{u}^\mu(x)$ are both

$$J = \det\,\Big(\frac{\partial\tau(x)}{\partial x^\mu}, \frac{\partial\sigma^r(x)}{\partial x^\mu}\Big) \neq 0. \tag{5.3}$$

By using

$$\frac{\partial\sigma^A(x)}{\partial x^\nu}\,\frac{\partial x^\mu}{\partial\sigma^A}(\sigma(x)) = \eta^\mu_\nu, \tag{5.4}$$

and Eq.(5.1) we get the desired result

$$\tilde{u}^\mu(x) = \tilde{u}^\nu(x)\,\frac{\partial\sigma^A(x)}{\partial x^\nu}\,\frac{\partial x^\mu}{\partial\sigma^A}(\sigma(x)) = \Big(\tilde{u}^\nu(x)\,\frac{\partial\tau(x)}{\partial x^\nu}\Big)\,\frac{\partial z^\mu(\tau,\vec{\sigma})}{\partial\tau} = \frac{z^\mu_\tau(\tau,\vec{\sigma})}{\sqrt{\boldsymbol{\epsilon}\,g_{\tau\tau}(\tau,\vec{\sigma})}}. \tag{5.5}$$

Given a unit time-like vector field $\tilde{u}^\mu(x) = u^\mu(\tau,\vec{\sigma})$ such that $u^\mu(\tau,\vec{\sigma}) \rightarrow_{|\vec{\sigma}|\rightarrow\infty} n^\mu(\tau)$ and $\frac{\partial u^\mu(\tau,\vec{\sigma})}{\partial\sigma^r} \rightarrow_{|\vec{\sigma}|\rightarrow\infty} 0$, to find the embeddings $z^\mu(\tau,\vec{\sigma})$ we must integrate the equation

$$\frac{\partial z^\mu(\tau,\vec{\sigma})}{\partial\tau} = f(\tau,\vec{\sigma})\,u^\mu(\tau,\vec{\sigma}), \qquad u^2(\tau,\vec{\sigma}) = \boldsymbol{\epsilon}, \tag{5.6}$$

where $f(\tau,\vec{\sigma})$ is an integrating factor.

Since Eq.(5.6) implies $\boldsymbol{\epsilon}\,g_{\tau\tau}(\tau,\vec{\sigma}) = f^2(\tau,\vec{\sigma}) > 0$, the only restrictions on the integrating factor are:

i) it must never vanish;

ii) $f(\tau,\vec{\sigma}) \rightarrow_{|\vec{\sigma}|\rightarrow\infty} f(\tau)$ finite.

The integration of Eq.(5.2) gives

$$z^\mu(\tau,\vec{\sigma}) \;=\; g^\mu(\vec{\sigma}) + \int_o^\tau d\tau_1\, f(\tau_1,\vec{\sigma})\,u^\mu(\tau_1,\vec{\sigma}),$$

$$\Downarrow$$

$$z^\mu_r(\tau,\vec{\sigma}) \;=\; \partial_r\, g(\vec{\sigma}) + \int_o^\tau d\tau_1\,\partial_r\,[f(\tau_1,\vec{\sigma})\,u^\mu(\tau,\vec{\sigma})],$$

time-like with $\tilde{u}^\mu(x)\,\frac{\partial\tau(x)}{\partial x^\mu} \neq 0$, so that we get $\alpha = 0$. We remain with the equations $\beta_r\,\frac{\partial\sigma^r(x)}{\partial x^\mu} = 0$, which imply $\beta_r = 0$ since the $\frac{\partial\sigma^r(x)}{\partial x^\mu}$ are independent by construction.

$$g_{\tau r}(\tau,\vec{\sigma}) \;=\; f(\tau,\vec{\sigma})\, u_\mu(\tau,\vec{\sigma}) \Big[\partial_r\, g(\vec{\sigma}) + \int_o^\tau d\tau_1\, \partial_r\, [f(\tau_1,\vec{\sigma})\, u^\mu(\tau,\vec{\sigma})]\Big]$$

$$\rightarrow_{|\vec{\sigma}|\rightarrow\infty} f(\tau)\, n_\mu(\tau) \Big[\lim_{|\vec{\sigma}|\rightarrow\infty} \partial_r\, g(\vec{\sigma})\Big], \qquad (5.7)$$

where $g(\vec{\sigma})$ is arbitrary and we have assumed that the integrating factor satisfies $\partial_r\, f(\tau,\vec{\sigma}) \rightarrow_{|\vec{\sigma}|\rightarrow\infty} 0$.

For the sake of simplicity let us choose $g(\vec{\sigma}) = \epsilon^\mu_r\, \sigma^r$ with the constant 4-vectors ϵ^μ_r belonging to an orthonormal tetrad ϵ^μ_A. Then $g_{\tau r}(\tau,\vec{\sigma})$ has the finite limit $f(\tau)\, n_\mu(\tau)\, \epsilon^\mu_r$.

With this choice for $g(\vec{\sigma})$ we get

$$z^\mu_r(\tau,\vec{\sigma}) \;=\; [\delta_{rs} + \alpha_{rs}(\tau,\vec{\sigma})]\, \epsilon^\mu_s + \beta_r(\tau,\vec{\sigma})\, \epsilon^\mu_\tau,$$

$$\alpha_{rs}(\tau,\vec{\sigma}) = \int_o^\tau d\tau_1\, \partial_r\, [f(\tau_1,\vec{\sigma})\, \epsilon_{s\mu}\, u^\mu(\tau_1,\vec{\sigma})],$$

$$\beta_r(\tau,\vec{\sigma}) = \int_o^\tau d\tau_1\, \partial_r\, [f(\tau_1,\vec{\sigma})\, \epsilon_{\tau\mu}\, u^\mu(\tau_1,\vec{\sigma})]. \qquad (5.8)$$

Since $u^\mu(\tau,\vec{\sigma})$ and ϵ^μ_τ are future time-like $[\boldsymbol{\epsilon}\, u^o(\tau,\vec{\sigma}) > 0,\ \boldsymbol{\epsilon}\, \epsilon^o_\tau > 0]$, we have $u^\mu(\tau,\vec{\sigma}) = \boldsymbol{\epsilon}\, a(\tau,\vec{\sigma})\, \epsilon^\mu_\tau + b_r(\tau,\vec{\sigma})\, \epsilon^\mu_r$ with $a(\tau,\vec{\sigma}) > 0$ and without zeroes.

Let us determine the integrating factor $f(\tau,\vec{\sigma})$ by requiring $\beta_r(\tau,\vec{\sigma}) = 0$ as a consequence of the equation

$$
\begin{aligned}
0 &= \boldsymbol{\epsilon}\, \partial_r \left[f(\tau,\vec{\sigma})\, \epsilon_{\tau\mu}\, u^\mu(\tau,\vec{\sigma}) \right] = f(\tau,\vec{\sigma})\, \partial_r\, a(\tau,\vec{\sigma}) + \partial_r\, f(\tau,\vec{\sigma})\, a(\tau,\vec{\sigma}),\\
&\Downarrow\\
f(\tau,\vec{\sigma}) &= e^{c(\tau)}\, a(\tau,\vec{\sigma}),\\
z^\mu_r(\tau,\vec{\sigma}) &= [\delta_{rs} + \alpha_{rs}(\tau,\vec{\sigma})]\, \epsilon^\mu_s,\\
\alpha_{rs}(\tau,\vec{\sigma}) &= \int_o^\tau d\tau_1\, e^{c(\tau_1)}\, \partial_r \left[a(\tau_1,\vec{\sigma})\, \epsilon_{s\mu}\, u^\mu(\tau_1,\vec{\sigma}) \right],\\
g_{rs}(\tau,\vec{\sigma}) &= -\boldsymbol{\epsilon} \Big(\delta_{rs} + \alpha_{rs}(\tau,\vec{\sigma}) + \alpha_{sr}(\tau,\vec{\sigma}) + \sum_u \alpha_{ru}(\tau,\vec{\sigma})\, \alpha_{su}(\tau,\vec{\sigma}) \Big).
\end{aligned} \tag{5.9}
$$

Let us choose the arbitrary function $C(\tau) = e^{c(\tau)}$ so small that $|\alpha_{rs}(\tau,\vec{\sigma})| << 1$ for every r, s, τ, $\vec{\sigma}$, so that all the conditions on $g_{rs}(\tau,\vec{\sigma})$ from Eqs.(1.1) are satisfied.

In conclusion given an arbitrary congruence of time-like world-lines, described by a vector field $\tilde{u}^\mu(x)$, an embedding defining a good notion of simultaneity is $[x^\mu(\tau) \stackrel{def}{=} z^\mu(\tau,\vec{0})]$

$$
\begin{aligned}
z^\mu(\tau,\vec{\sigma}) &= \epsilon^\mu_r\, \sigma^r + \int_o^\tau d\tau_1\, C(\tau_1)\, \epsilon_{\tau\nu}\, u^\nu(\tau_1,\vec{\sigma})\, u^\mu(\tau_1,\vec{\sigma}) =\\
&= x^\mu(\tau) + \epsilon^\mu_r\, \sigma^r + \int_o^\tau d\tau_1\, C(\tau_1)\, \epsilon_{\tau\nu} \Big[u^\nu(\tau_1,\vec{\sigma})\, u^\mu(\tau_1,\vec{\sigma}) - u^\nu(\tau,\vec{0})\, u^\mu(\tau,\vec{0}) \Big],
\end{aligned} \tag{5.10}
$$

for sufficiently small $C(\tau)$. Here ϵ^μ_A is an arbitrary orthonormal tetrad.

VI. APPLICATIONS.

In this Section we shall apply the 3+1 point of view to the description of GPS, to the problem of the rotating disk and of the Sagnac effect, to the determination of the time delay for light propagation between an Earth station and a satellite and finally to Maxwell theory.

A. The Global Positioning System and the Determination of a Set of Radar Coordinates.

In Eqs.(4.1) we gave a family of embeddings $x^\mu = z^\mu(\tau, \vec{\sigma})$ defining possible notions of simultaneity, i.e. admissible 3+1 splittings of Minkowski space-time with foliations with space-like hyper-planes Σ_τ as leaves, to be associated to the world-line $x^\mu(\tau)$ of an arbitrary time-like observer γ, chosen as origin of the 3-coordinates on each simultaneity leaf Σ_τ, i.e. $x^\mu(\tau) = z^\mu(\tau, \vec{0})$. The space-like hyper-planes Σ_τ are not orthogonal to γ: if $l^\mu = \epsilon^\mu_\tau$ is the normal to Σ_τ we have $l_\mu \frac{\dot{x}^\mu(\tau)}{\sqrt{\boldsymbol{\epsilon}\, \dot{x}^2(\tau)}} \neq \boldsymbol{\epsilon}$ except in the limiting case of an inertial observer with 4-velocity proportional to l^μ.

If τ is the scalar coordinate-time of the foliation, the proper time of the standard atomic clock C of γ will be defined by $d\mathcal{T}_\gamma = \sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{0})}\, d\tau$ $[x^\mu(\tau) = \tilde{x}^\mu(\mathcal{T}_\gamma)]$. This defines $\mathcal{T}_\gamma = \mathcal{F}_\gamma(\tau)$ as a monotonic function of τ, whose inverse will be denoted $\tau = \mathcal{G}(\mathcal{T}_\gamma)$. Moreover, we make a conventional choice of a tetrad ${}_{(\gamma)}E^\mu_A(\tau)$ associated to γ with ${}_{(\gamma)}E^\mu_\tau(\tau) = \frac{\dot{x}^\mu(\tau)}{\sqrt{\boldsymbol{\epsilon}\, \dot{x}^2(\tau)}}$.

Let us consider a set of N arbitrary time-like world-lines $x^\mu_i(\tau)$, $i = 1, .., N$, associated to observers γ_i, so that γ and the γ_i's can be imagined to be the world-lines of $N+1$ spacecrafts (like in GPS [60]) with γ chosen as a reference world-line. Each of the world-lines γ_i will have an associated standard atomic clock C_i and a conventional tetrad ${}_{(\gamma_i)}E^\mu_A(\tau)$.

To compare the distant clocks C_i with C in the chosen notion of simultaneity, we define the 3-coordinates $\vec{\eta}_i(\tau)$ of the γ_i

$$x^\mu_i(\tau) \stackrel{def}{=} z^\mu(\tau, \vec{\eta}_i(\tau)). \tag{6.1}$$

Then the proper times $\mathcal{T}_{\gamma_i}$ of the clocks C_i will be expressed in terms of the scalar coordinate time τ of the chosen simultaneity as

$$d\mathcal{T}_{\gamma_i} = \sqrt{\boldsymbol{\epsilon}\, \Big[g_{\tau\tau}(\tau, \vec{\eta}_i(\tau)) + 2 g_{\tau r}(\tau, \vec{\eta}_i(\tau))\, \dot{\eta}^r(\tau) + g_{rs}(\tau, \vec{\eta}_i(\tau))\, \dot{\eta}^r(\tau)\, \dot{\eta}^s(\tau) \Big]}\, d\tau, \tag{6.2}$$

so that with this notion of simultaneity the proper times $\mathcal{T}_{\gamma_i}$ are connected to the proper time $\mathcal{T}_\gamma$ by the following relations

$$
d\mathcal{T}_{\gamma_i} = \sqrt{\frac{g_{\tau\tau}(\tau,\vec{\eta}_i(\tau)) + 2g_{\tau r}(\tau,\vec{\eta}_i(\tau))\,\dot{\eta}_i^r(\tau) + g_{rs}(\tau,\vec{\eta}_i(\tau))\,\dot{\eta}_i^r(\tau)\,\dot{\eta}_i^s(\tau)}{g_{\tau\tau}(\tau,\vec{0})}}\Bigg|_{\tau=\mathcal{G}(\mathcal{T}_\gamma)} d\mathcal{T}_\gamma. \quad (6.3)
$$

This determines the synchronization of the $N+1$ clocks once we have expressed the 3-coordinates $\vec{\eta}_i(\tau)$ in terms of the given world-lines $x^\mu(\tau)$, $x_i^\mu(\tau)$ and of the embedding (4.1). From the definition

$$
x_i^\mu(\tau) = z^\mu(\tau,\vec{\eta}_i(\tau)) = x^\mu(\tau) + \epsilon^\mu_r\, R^r{}_s(\tau,|\vec{\eta}_i(\tau)|)\,\eta_i^s(\tau), \quad (6.4)
$$

we get $[|\vec{\eta}_i(\tau)| \overset{def}{=} \sqrt{\delta_{rs}\,\eta_i^r(\tau)\,\eta_i^s(\tau)},\ \eta_i^r(\tau) = |\vec{\eta}_i(\tau)|\,\hat{n}_i^r(\tau),\ \delta_{rs}\,\hat{n}_i^r(\tau)\,\hat{n}_i^s(\tau) = 1]$

$$
\eta_i^u(\tau) = -\sum_w [R^{-1}(\tau,|\vec{\eta}_i(\tau)|)]^u{}_w\,\epsilon^\nu_w\,[x_{i\nu}(\tau) - x_\nu(\tau)]. \quad (6.5)
$$

Then, if we put the solution

$$
|\vec{\eta}_i(\tau)| = F_i\Big[\epsilon^\mu_r\,\Big(x_{i\mu}(\tau) - x_\mu(\tau)\Big)\Big], \quad (6.6)
$$

of the equations

$$
\begin{aligned}
|\vec{\eta}_i(\tau)|^2 \;=\; \delta_{rs} \sum_{mn} &[R^{-1}(\tau,|\vec{\eta}_i(\tau)|)]^r{}_m\,[R^{-1}(\tau,|\vec{\eta}_i(\tau)|)]^s{}_n \\
&\epsilon^\mu_m\,[x_{i\mu}(\tau) - x_\mu(\tau)]\,\epsilon^\nu_n\,[x_{i\nu}(\tau) - x_\nu(\tau)], \quad (6.7)
\end{aligned}
$$

into Eqs.(6.5), we obtain the looked for expression of the 3-coordinates $\vec{\eta}_i(\tau)$

$$
\eta_i^u(\tau) = -\sum_m \Big[R^{-1}(\tau, F_i[\epsilon^\alpha_w\,(x_{i\alpha}(\tau) - x_\alpha(\tau))])\Big]^u{}_m\,\epsilon^\nu_m\,[x_{i\nu}(\tau) - x_\nu(\tau)]. \quad (6.8)
$$

We will now define *an operational method to build a grid of radar 4-coordinates* associated with the arbitrarily given time-like world-line $x^\mu(\tau)$ of the spacecraft γ and with an admissible embedding $z^\mu(\tau,\vec{\sigma})$ (we use Eq.(4.1) as an example), by using light signals emitted by γ and reflected towards γ from the other spacecrafts γ_i. This will justify the name radar

4-coordinates and will show how the simultaneity convention (4.1) deviates from Einstein's convention.

To this end, given an embedding $x^\mu = z^\mu(\tau, \vec{\sigma})$ of the family (4.1) and one of its simultaneity leaves Σ_τ with the point Q of 4-coordinates $x^\mu(\tau)$ on γ as origin of the 3-coordinates $\vec{\sigma}$, let us consider a point P on Σ_τ with coordinates $z^\mu(\tau, \vec{\sigma})$ (for $\vec{\sigma} = \vec{\eta}_i(\tau)$ it corresponds to the spacecraft γ_i). We want to express the adapted 4-coordinates $\tau = \tau(P)$, $\vec{\sigma} = \vec{\sigma}(P)$ of P in terms of data on the world-line γ corresponding to the emission of a light signal in Q_- at $\tau_- < \tau$ and to its reception in Q_+ at $\tau_+ > \tau$ after reflection at P.

Let $x^\mu(\tau_-)$ be the intersection of the world-line γ with the past light-cone through P and $x^\mu(\tau_+)$ the intersection with the future light-cone through P. To find $\tau_\pm$ we have to solve the equations $\Delta^2_\pm = [x^\mu(\tau_\pm) - z^\mu(\tau, \vec{\sigma})]^2 = 0$ with $\Delta^\mu_\pm = x^\mu(\tau_\pm) - z^\mu(\tau, \vec{\sigma})$. We are interested in the solutions $\Delta^o_+ = |\vec{\Delta}_+|$ and $\Delta^o_- = -|\vec{\Delta}_-|$. Let us remark that on the simultaneity surfaces Σ_τ we have $x^o(\tau) \neq z^o(\tau, \vec{\sigma})$ for the Cartesian coordinate times.

Let us show that the adapted coordinates τ and $\vec{\sigma}$ of the event P with Cartesian 4-coordinates $z^\mu(\tau, \vec{\sigma})$ in an inertial system can be determined in terms of the *emission scalar time* τ_- of the light signal, the *emission unit 3-direction* $\hat{n}_{(\tau_-)}(\theta_{(\tau_-)}, \phi_{(\tau_-)})$ [so that $\triangle^\mu_- = |\vec{\triangle}_-|\,(-\boldsymbol{\epsilon}; \hat{n}_{(\tau_-)})$] and the *reception scalar time* τ_+ registered by the observer γ with world-line $x^\mu(\tau)$. These data are usually given in terms of the proper time $\mathcal{T}(\tau)$ of the observer γ by using $d\mathcal{T} = \sqrt{\boldsymbol{\epsilon}\, g_{\tau\tau}(\tau, \vec{0})}\, d\tau$.

Let us introduce the following parametrization by using Eqs.(4.1)

$$
\begin{aligned}
z^\mu(\tau, \vec{\sigma}) &= x^\mu(\tau) + \epsilon^\mu_r\, R^r{}_s(\tau, \sigma)\, \sigma^s \stackrel{def}{=} \\
&\stackrel{def}{=} \Big[\xi_l(\tau, \vec{\sigma})\, l^\mu + \xi^r(\tau, \vec{\sigma})\, \epsilon^\mu_r\Big] = \\
&= \Big[x_l(\tau)\, l^\mu + \sum_r \left[x^r_\epsilon(\tau) + \zeta^r(\tau, \vec{\sigma})\right] \epsilon^\mu_r\Big], \\
\xi_l(\tau, \vec{\sigma}) &= \boldsymbol{\epsilon}\, z_\mu(\tau, \vec{\sigma})\, l^\mu = \boldsymbol{\epsilon}\, x_\mu(\tau)\, l^\mu = x_l(\tau),
\end{aligned}
$$

$$\xi^r(\tau,\vec{\sigma}) = \boldsymbol{\epsilon}\, z_\mu(\tau,\vec{\sigma})\,\epsilon^\mu_r = x^r_\epsilon(\tau) + \zeta^r(\tau,\vec{\sigma}),$$

$$x^r_\epsilon(\tau) = \boldsymbol{\epsilon}\, x_\mu(\tau)\,\epsilon^\mu_r, \qquad \zeta^r(\tau,\vec{\sigma}) = R^r{}_s(\tau,\sigma)\,\sigma^s \rightarrow_{\sigma\rightarrow\infty} \sigma^r. \tag{6.9}$$

Then the two equations $\triangle^2_\pm = [x^\mu(\tau_\pm) - z^\mu(\tau,\vec{\sigma})]^2 = \boldsymbol{\epsilon}\,([x_l(\tau_\pm) - x_l(\tau)]^2 - [\vec{x}_\epsilon(\tau_\pm) - \vec{x}_\epsilon(\tau) - \vec{\zeta}(\tau,\vec{\sigma})]^2) = 0$ can be rewritten in the form

$$x_l(\tau_+) = x_l(\tau) + |\vec{\triangle}_+| = x_l(\tau) + |\vec{x}_\epsilon(\tau_+) - \vec{\xi}(\tau,\vec{\sigma})|,$$
$$|\vec{\triangle}_+| = |\vec{x}_\epsilon(\tau_+) - \vec{\xi}(\tau,\vec{\sigma})| \rightarrow_{\sigma\rightarrow\infty} |\vec{x}_\epsilon(\tau_+) - \vec{x}_\epsilon(\tau) - \vec{\sigma}|,$$

$$x_l(\tau_-) = x_l(\tau) - |\vec{\triangle}_-| = x_l(\tau) - |\vec{x}_\epsilon(\tau_-) - \vec{\xi}(\tau,\vec{\sigma})|,$$
$$|\vec{\triangle}_-| = |\vec{x}_\epsilon(\tau_-) - \vec{\xi}(\tau,\vec{\sigma})| \rightarrow_{\sigma\rightarrow\infty} |\vec{x}_\epsilon(\tau_-) - \vec{x}_\epsilon(\tau) - \vec{\sigma}|. \tag{6.10}$$

It can be shown [100] that, if no observer is allowed to become a Rindler observer [99], then each equation admits a unique [37] solution $\tau_\pm = T_\pm(\tau,\vec{\sigma})$.

Therefore the following four data measured by the observer γ

$$\tau_\pm = T_\pm(\tau,\vec{\sigma}),$$

[37] If we introduce the function

$$g_\pm(y) = x_l(y) - x_l(\tau) \pm |\vec{x}_\epsilon(y) - \vec{\xi}(\tau,\vec{\sigma})|,$$

Eqs. (6.10) are equivalent to $g_\pm(y) = 0$. The solution is unique because the functions $g_\pm(y)$ are decreasing in y, since we have

$$\frac{dg_\pm(y)}{dy} = -v_l(y) \pm \sum_r v_r(y)\,\frac{x^r_\epsilon(y) - \xi^r(\tau,\vec{\sigma})}{|\vec{x}_\epsilon(y) - \vec{\xi}(\tau,\vec{\sigma})|}.$$

Using Eq.(4.11) in the form

$$\sum_r v_r(y)\,\frac{x^r_\epsilon(y) - \xi^r(\tau,\vec{\sigma})}{|\vec{x}_\epsilon(y) - \vec{\xi}(\tau,\vec{\sigma})|} \leq |\vec{v}(y)| < v_l(y),$$

we get $\frac{dg_\pm(y)}{dy} < 0$, since $v_l(y) > 0$.

$$\hat{n}_{(\tau_-)}(\theta_{(\tau_-)}, \phi_{(\tau_-)}) \;=\; \Big(\sin\,\theta_{(\tau_-)}\,\sin\,\phi_{(\tau_-)}, \sin\,\theta_{(\tau_-)}\,\cos\,\phi_{(\tau_-)}, \cos\,\theta_{(\tau_-)}\Big) =$$

$$= \frac{\vec{\triangle}_-}{|\vec{\triangle}_-|} = \frac{\vec{x}_\epsilon(\tau_-) - \vec{x}_\epsilon(\tau) - \vec{\zeta}(\tau,\vec{\sigma})}{|\vec{x}_\epsilon(\tau_-) - \vec{x}_\epsilon(\tau) - \vec{\zeta}(\tau,\vec{\sigma})|}\Bigg|_{\tau_-=T_-(\tau,\vec{\sigma})} = \hat{m}(\tau,\vec{\sigma}), \quad (6.11)$$

can be inverted to get the adapted coordinates $\tau(P)$, $\vec{\sigma}(P)$ of the event P with 4-coordinates $z^\mu(\tau,\vec{\sigma})$ in terms of the data (Einstein's convention for the radar time would be $\mathcal{E} = \frac{1}{2}$)

$$\begin{aligned}
\tau(P) \;&=\; \tau(\tau_-, \hat{n}_{(\tau_-)}, \tau_+) \stackrel{def}{=} \tau_- + \mathcal{E}(\tau_-, \hat{n}_{(\tau_-)}, \tau_+)\,[\tau_+ - \tau_-],\\
\vec{\sigma}(P) \;&=\; \vec{\mathcal{G}}(\tau_-, \hat{n}_{(\tau_-)}, \tau_+) \rightarrow_{\tau_+\rightarrow\tau_-} 0.
\end{aligned} \qquad (6.12)$$

Let us remark that

i) for $x^\mu(\tau) = \tau\, l^\mu$ (inertial observer with world-line *orthogonal* to Σ_τ; $\vec{x}_\epsilon(\tau) = 0$) we get the Einstein's convention for the radar time, because we have

$$\begin{aligned}
\tau_\pm \;&=\; \tau \pm |\vec{\zeta}(\tau,\vec{\sigma})|, \qquad \tau = \frac{1}{2}\,(\tau_+ + \tau_-), \qquad \sigma = |\vec{\zeta}(\tau,\vec{\sigma})| = \frac{1}{2}\,(\tau_+ - \tau_-),\\
\mathcal{E} \;&=\; \frac{1}{2}, \qquad \zeta^r(\tau,\vec{\sigma}) = -\frac{1}{2}\,(\tau_+ - \tau_-)\,\hat{n}^r_{(\tau_-)},\\
\sigma^r \;&=\; \mathcal{G}^r = \frac{1}{2}\,(\tau_+ - \tau_-)\,(R^{-1})^r{}_s(\frac{\tau_+ + \tau_-}{2}, \frac{\tau_+ - \tau_-}{2})\,\hat{n}^s_{(\tau_-)};
\end{aligned}$$

ii) for $x^\mu(\tau) = \tau\,[l^\mu + \epsilon^\mu_r\, a^r]$ (inertial observer with world-line *non-orthogonal* to Σ_τ; $\vec{x}_\epsilon(\tau) = \tau\,\vec{a}$), after some straightforward calculations, we get

$$\begin{aligned}
\tau_\pm \;&=\; \tau + \frac{1}{1-\vec{a}^2}\Big[-\vec{a}\cdot\vec{\zeta}(\tau,\vec{\sigma}) \pm \sqrt{(\vec{a}\cdot\vec{\zeta}(\tau,\vec{\sigma}))^2 + (1-\vec{a}^2)\,\sigma^2}\Big],\\
\tau \;&=\; \frac{1}{2}\Big[\tau_+ + \tau_- + \frac{\tau_+ - \tau_-}{1-\vec{a}^2}\sqrt{\frac{\vec{a}^2 + \vec{a}\cdot\hat{n}_{(\tau_-)}}{1 + \vec{a}^2 - \vec{a}^4 + (3 - 2\,\vec{a}^2)\,\vec{a}\cdot\hat{n}_{(\tau_-)}}}\Big],\\
\mathcal{E} \;&=\; \frac{1}{2}\Big[1 + \frac{1}{1-\vec{a}^2}\sqrt{\frac{\vec{a}^2 + \vec{a}\cdot\hat{n}_{(\tau_-)}}{1 + \vec{a}^2 - \vec{a}^4 + (3 - 2\,\vec{a}^2)\,\vec{a}\cdot\hat{n}_{(\tau_-)}}}\Big],
\end{aligned}$$

$$
\begin{aligned}
\sigma &= |\vec{\zeta}(\tau,\vec{\sigma})| = \frac{1}{2}\,(\tau_+ - \tau_-)\,\sqrt{\frac{1+\vec{a}^2+2\,\vec{a}\cdot\hat{n}_{(\tau_-)}}{1+\vec{a}^2-\vec{a}^4+(3-2\,\vec{a}^2)\,\vec{a}\cdot\hat{n}_{(\tau_-)}}},\\
\frac{\zeta^r(\tau,\vec{\sigma})}{|\vec{\zeta}(\tau,\vec{\sigma})|} &= -\frac{\sqrt{\vec{a}^2+\vec{a}\cdot\hat{n}_{(\tau_-)}}+\sqrt{1+\vec{a}^2-\vec{a}^4+(3-2\,\vec{a}^2)\,\vec{a}\cdot\hat{n}_{(\tau_-)}}}{\sqrt{1+\vec{a}^2+2\,\vec{a}\cdot\hat{n}_{(\tau_-)}}}\,\frac{a^r+\hat{n}^r_{(\tau_-)}}{1-\vec{a}^2},\\
\sigma^r &= \mathcal{G}^r = -\frac{1}{2}\,(\tau_+-\tau_-)\,\Big(1+\sqrt{\frac{\vec{a}^2+\vec{a}\cdot\hat{n}_{(\tau_-)}}{1+\vec{a}^2-\vec{a}^4+(3-2\,\vec{a}^2)\,\vec{a}\cdot\hat{n}_{(\tau_-)}}}\Big)\\
&\quad (R^{-1})^r{}_s\Big(\frac{1}{2}\,\Big[\tau_++\tau_-+\frac{\tau_+-\tau_-}{1-\vec{a}^2}\,\sqrt{\frac{\vec{a}^2+\vec{a}\cdot\hat{n}_{(\tau_-)}}{1+\vec{a}^2-\vec{a}^4+(3-2\,\vec{a}^2)\,\vec{a}\cdot\hat{n}_{(\tau_-)}}}\Big],\\
&\quad \frac{1}{2}\,(\tau_+-\tau_-)\,\sqrt{\frac{1+\vec{a}^2+2\,\vec{a}\cdot\hat{n}_{(\tau_-)}}{1+\vec{a}^2-\vec{a}^4+(3-2\,\vec{a}^2)\,\vec{a}\cdot\hat{n}_{(\tau_-)}}}\Big)\quad \frac{a^s+\hat{n}^s_{(\tau_-)}}{1-\vec{a}^2};
\end{aligned}
$$

iii) for non-inertial trajectories $x^\mu(\tau) = f(\tau)\,l^\mu + \epsilon^\mu_r\,g^r(\tau)$ $[\boldsymbol{\epsilon}\,[\dot{f}^2(\tau) - \sum_r \dot{g}^r(\tau)\,\dot{g}^r(\tau)] > 0]$ the evaluation of $\mathcal{E}$ and $\vec{\mathcal{G}}$ cannot be done analytically, but only numerically.

Let us now consider an infinitesimal displacement $\delta z^\mu = z^\mu(\tau+\delta\tau,\vec{\sigma}+\delta\vec{\sigma}) - z^\mu(\tau,\vec{\sigma})$ of P on Σ_τ to P' on $\Sigma_{\tau+\delta\tau}$. The event P' will receive light signals from the event $Q(\tau_-+\delta\tau_-)$ on γ and will reflect them towards the event $Q(\tau_++\delta\tau_+)$ on γ. Now, using $\Delta^2_\pm = 0$, we have $\Delta^{'\mu}_\pm = \Delta^\mu_\pm + \dot{x}^\mu(\tau_\pm)\,\delta\tau_\pm - \delta z^\mu$ and $\Delta^{'2}_\pm = 2\,\Delta^\mu_\pm\,[\dot{x}_\mu(\tau_\pm)\,\delta\tau_\pm - \delta z_\mu] + (higher\, order\, terms)$. As a consequence we get (see Ref.[44])

$$
\begin{aligned}
\frac{\partial\tau_\pm}{\partial z^\mu} &= \frac{\Delta_{\pm\,\mu}}{\Delta_\pm\cdot\dot{x}(\tau_\pm)},\\
with &\qquad \boldsymbol{\epsilon}\,\Delta_+\cdot\Delta_- < 0,\quad \boldsymbol{\epsilon}\,\dot{x}(\tau_+)\cdot\Delta_+ > 0,\quad \boldsymbol{\epsilon}\,\dot{x}(\tau_-)\cdot\Delta_- < 0. \qquad (6.13)
\end{aligned}
$$

Since $\frac{\partial\tau(P)}{\partial z^\mu}$ is a time-like 4-vector orthogonal to Σ_τ, it must be proportional to the normal l^μ to the space-like hyper-planes of the foliation (4.1) till now considered. For a general admissible foliation we have (from $\triangle^2_- = 0$ we get $\triangle_-\cdot\frac{\partial\triangle_-}{\partial z^\mu} = 0$ and then $\triangle^\mu_-\,\frac{\partial\hat{n}_{\tau_-}}{\partial z^\mu} = 0$; instead in general $\triangle^\mu_+\,\frac{\partial\hat{n}_{\tau_-}}{\partial z^\mu} \neq 0$)

$$
\begin{aligned}
\frac{\partial \tau(P)}{\partial z^\mu} &= \Big[\mathcal{E} + (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial \tau_+}\Big]\,\frac{\partial \tau_+}{\partial z^\mu} + \\
&+ \Big[1 - \mathcal{E} + (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial \tau_-}\Big]\,\frac{\partial \tau_-}{\partial z^\mu} + \\
&+ (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial\,\hat{n}_{(\tau_-)}}\,\frac{\partial \hat{n}_{(\tau_-)}}{\partial z^\mu} = \\
&= \Big[\mathcal{E} + (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial \tau_+}\Big]\,\frac{\Delta_{+\,\mu}}{\Delta_+ \cdot \dot{x}(\tau_+)} + \\
&+ \Big[1 - \mathcal{E} + (\tau_+ - \tau_-)\Big(\frac{\partial \mathcal{E}}{\partial \tau_-} + \frac{\partial \mathcal{E}}{\partial\,\hat{n}_{(\tau_-)}}\,\frac{\partial \hat{n}_{(\tau_-)}}{\partial \tau_-}\Big)\Big]\,\frac{\Delta_{-\,\mu}}{\Delta_- \cdot \dot{x}(\tau_-)} + \\
&+ (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial\,\hat{n}_{(\tau_-)}}\,\frac{\partial \hat{n}_{(\tau_-)}}{\partial z^\mu},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\epsilon}\Big(\frac{\partial \tau(P)}{\partial z^\mu}\Big)^2 &= \boldsymbol{\epsilon}\,\frac{\Delta_+ \cdot \Delta_-}{\Delta_+ \cdot \dot{x}(\tau_+)\,\Delta_- \cdot \dot{x}(\tau_-)}\,\Big[\mathcal{E} + (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial \tau_+}\Big] \\
&\Big[1 - \mathcal{E} + (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial \tau_-}\Big] + (\tau_+ - \tau_-)^2\Big(\frac{\partial \mathcal{E}}{\partial\,\hat{n}_{(\tau_-)}}\Big)^2\Big(\frac{\partial \hat{n}_{(\tau_-)}}{\partial z^\mu}\Big)^2 + \\
&+ 2\,(\tau_+ - \tau_-)\Big[\mathcal{E} + (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial \tau_+}\Big]\,\frac{\partial \mathcal{E}}{\partial\,\hat{n}_{(\tau_-)}}\,\frac{\triangle_+ \cdot \frac{\partial \hat{n}_{(\tau_-)}}{\partial z^\mu}}{\triangle_+ \cdot \dot{x}(\tau_+)} > 0,
\end{aligned}
$$

$$
for\,every\;\; \tau_-, \theta_{(\tau_-)}, \phi_{(\tau_-)}, \tau_+. \tag{6.14}
$$

This is the condition on the function $\mathcal{E}(\tau_-, \hat{n}_{(\tau_-)}, \tau_+)$ to have an admissible foliation.

Since $\boldsymbol{\epsilon}\,\frac{\Delta_+ \cdot \Delta_-}{\Delta_+ \cdot \dot{x}(\tau_+)\,\Delta_- \cdot \dot{x}(\tau_-)} > 0$, in the special case $\frac{\partial \mathcal{E}}{\partial \hat{n}_{(\tau_-)}} = 0$ it must be

$$
\Big[\mathcal{E} + (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial \tau_+}\Big]\,\Big[1 - \mathcal{E} + (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial \tau_-}\Big] > 0,
$$

$$
\Downarrow
$$

$$
\mathcal{E} + (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial \tau_+} \neq 0, \qquad 1 - \mathcal{E} + (\tau_+ - \tau_-)\,\frac{\partial \mathcal{E}}{\partial \tau_-} \neq 0. \tag{6.15}
$$

Finally the functions $\mathcal{E}$ and $\vec{\mathcal{G}}$ must have a finite limit for $\tau_\pm \to \pm\infty$, i.e. at spatial infinity on Σ_τ.

Given the world-line $x^\mu(\tau)$ of an observer γ and four functions $0 < \mathcal{E}(\tau_-, \hat{n}_{(\tau_-)}, \tau_+) < 1$ and $\vec{\mathcal{G}}(\tau_-, \hat{n}_{(\tau_-)}, \tau_+) \rightarrow_{\tau_+ \rightarrow \tau_-} 0$, with $\mathcal{E}$ satisfying Eq.(6.14), we can build the admissible adapted 4-coordinates τ, $\vec{\sigma}$ of a γ-dependent notion of simultaneity, because Eqs.(6.14) and (6.15) ensure that the surfaces Σ_τ are space-like since their normal $\frac{\partial \tau(P)}{\partial z^\mu}$ is everywhere time-like.

To reconstruct the embedding associated to this notion of simultaneity we must invert the Jacobian matrix $b^A_\mu = \frac{\partial \sigma^A(P)}{\partial z^\mu}$ and find the matrix $b^\mu_A = \frac{\partial z^\mu(P)}{\partial \sigma^A}$ satisfying the conditions $b^A_\mu\, b^\mu_B = \delta^A_B$, $b^A_\mu\, b^\nu_A = \delta^\nu_\mu$. Then by integrating $\frac{\partial z^\mu(P)}{\partial \sigma^A}$ we get the associated embedding $x^\mu = z^\mu(\tau, \vec{\sigma})$. Let us remark that for $\sigma \rightarrow \infty$ b^μ_A must tend in a direction-independent way to the asymptotic tetrad ϵ^μ_A associated to the asymptotic space-like hyper-planes of any admissible foliation. As a consequence $b^A_\mu = \frac{\partial \sigma^A(P)}{\partial z^\mu}$ must tend to the asymptotic cotetrad ϵ^A_μ. This is a condition on the admissible functions $\mathcal{E}$ and $\vec{\mathcal{G}}$.

If we call $|\vec{\Delta}_-|$ the *light distance of* Q_- *on* γ *to* P and $|\vec{\Delta}_+|$ the *light distance of* P *to* Q_+ *on* γ (see Section 4 of Ref.[21]) we get the following two one-way velocities of light (with $c = 1$) in coordinates adapted to the given notion of simultaneity

$$
\begin{aligned}
c_- \,&=\, \frac{|\vec{\Delta}_-|}{\tau - \tau_-} = \frac{|\vec{\Delta}_-|}{\mathcal{E}\,(\tau_+ - \tau_-)} = \frac{2\,\eta\,|\vec{\Delta}|}{\mathcal{E}\,(\tau_+ - \tau_-)}, \qquad from\, Q_-\, to\, P,\\
c_+ \,&=\, \frac{|\vec{\Delta}_+|}{\tau_+ - \tau} = \frac{|\vec{\Delta}_+|}{(1-\mathcal{E})\,(\tau_+ - \tau_-)} = \frac{2\,(1-\eta)\,|\vec{\Delta}|}{(1-\mathcal{E})\,(\tau_+ - \tau_-)}, \qquad from\, P\, to\, Q_+,\\
&|\vec{\Delta}| \overset{def}{=} \frac{1}{2}\,(|\vec{\Delta}_+| + |\vec{\Delta}_-|), \qquad \eta \overset{def}{=} \frac{|\vec{\Delta}_-|}{|\vec{\Delta}_+| + |\vec{\Delta}_-|}.
\end{aligned}
\tag{6.16}
$$

If $c_\tau = \frac{2\,|\vec{\Delta}|}{\tau_+ - \tau_-}$ is the isotropic average round-trip τ-coordinate velocity of light, we get $c_+ = \frac{1-\eta}{1-\mathcal{E}}\, c_\tau$, $c_- = \frac{\eta}{\mathcal{E}}\, c_\tau$.

If $x^\mu(\tau)$ is a straight-line (inertial observer) we can adopt Einstein's convention $\mathcal{E} = \frac{1}{2}$, i.e. $\tau(P) = \frac{1}{2}\,(\tau_+ + \tau_-)$ and $|\vec{\sigma}| = |\vec{\mathcal{G}}| = \frac{1}{2}\,(\tau_+ - \tau_-)$ (hyper-planes orthogonal to the observer world-line). This implies $|\vec{\Delta}_+| = |\vec{\Delta}_-|$ and $\eta = \frac{1}{2}$.

Instead, if we ask $c_\tau = c_+ = c_-$, i.e. *isotropy* of light propagation, we get $\mathcal{E} = \eta$. This shows that once we have made a convention on *two* of the quantities *spatial distance*, *one-way speed of light* and *simultaneity*, the third one is automatically determined [21].

In general relativity on globally hyperbolic space-times, we can define in a similar way the allowed global notions of simultaneity and the allowed one-way velocities of test light.

Then the knowledge of the factor $\mathcal{E}$ associated to an allowed notion of simultaneity will allow an operational determination of the 4-coordinates $(\tau, \vec{\sigma})$ adapted to the chosen notion of simultaneity with simultaneity surfaces $\tau = const.$ as radar coordinates. This is a step towards implementing the operational definition of space-time proposed in Refs. [8, 9]. The lacking ingredient is an operational confrontation of the tetrads ${}_{(\gamma_i)}E^\mu_A(\tau)$ with the tetrad ${}_{(\gamma)}E^\mu_A(\tau)$ of the reference world-line: this would allow a determination of the 4-metric in the built radar 4-coordinates and a reconstruction of a finite region of space-time around the $N+1$ spacecrafts of the GPS type, whose trajectories are supposed known (for instance determined with the standard techniques of space navigation [101] controlled by a station on the Earth). See Refs.[102] for other approaches to GPS type coordinates.

However, as we shall comment in the Conclusions, in general relativity the admissible notions of simultaneity are *dynamically* determined by the Hamilton equations, equivalent to Einstein's equations, of the ADM canonical formulation of metric gravity.

B. The 3+1 Point of View on the Rotating Disk.

Let us now consider the 3+1 point of view about the problem of the *rotating disk*, which is still under debate after nearly a century of proposals for the resolution of the *Ehrenfest paradox* (see Ref.[45, 55] for a rich bibliography on the most relevant points of view on the rotating disk). A basic ambiguity in the formulation of the problem comes from the non-relativistic notion of a rigid (either geometrical or material) disk put in global rigid motion (this is possible due to the existence of arbitrary non-relativistic rigid rotating reference frames). At the relativistic level we have:

1) Rigid bodies do not exist. At best we can speak of *Born rigid motions* [103] and *Born reference frames* [38]. However Grøn [104] has shown that the acceleration phase of a material disk is not compatible with Born rigid motions. Moreover, we do not have a well formulated and accepted relativistic framework to discuss a relativistic elastic material disk (see Ref.[105] for a review), so that many statements in the literature cannot be checked with actual calculations. As a consequence most of the authors treating the rotating disk (either explicitly or implicitly) consider it as a geometrical entity, to be identified with a congruence of time-like world-lines (helices, see Ref.[106]) with non-zero vorticity [39]. But this corresponds to a model of material disk in which it is composed of a relativistic perfect fluid with zero pressure, i.e. to a relativistic dust contained in a cylindrical world-tube in Minkowski space-time (in the Cartesian 4-coordinates of an inertial system the restriction is $r = \sqrt{\sum_r (x^r)^2} \leq R$).

2) As we have shown in Section III, Eq.(3.2), relativistic rigid rotating reference frames do not exist. Therefore all the rotating reference frames appearing in the literature are only

[38] A reference frame or platform is *Born-rigid* [2] if in Eq.(2.1) the expansion Θ and the shear $\sigma_{\mu\nu}$ vanish, i.e. if the spatial distance between neighboring world-lines remains constant.

[39] I.e. non-surface-forming and therefore non-synchronizable. Therefore the observers associated to this congruence have neither a notion of global simultaneity nor a notion of instantaneous 3-space (since we do not have a preferred observer, we cannot use its local rest frame as 3-space like it was discussed in footnote 11: each observer will have a different local rest frame). As shown in Ref.[55] the only meaningful concept which can be defined is an *abstract relative 3-space*, i.e. the space whose points are the time-like world-lines of the congruence. Another problem is the definition of the *rods and clocks* of an observer of the congruence. As shown in Ref.[55] the two existing notions are: a) an *optical congruence* with the light null 4-geodesics approximated by 3-geodesics (used in Ref.[55]); b) a *congruence of Sevres meters* (i.e. a measurement of spatial distances with slowly transported rigid rods by definition not changing their length under acceleration; it was used by Einstein [10], Møller [12], Landau-Lifschitz [11]) with *free ends* (instead in Ref.[53] the rod was identified with a piece of the rim of the disk).

locally defined due to the horizon problem, so that the vector fields defining the relativistic frames are only defined on a sub-manifold Minkowski space-time containing the disk.

The 3+1 point of view looks at these problems in a different way and suggests the following re-formulation of the rotating disk. Let the disk be a relativistic isolated system (either a relativistic material body or a relativistic fluid or a relativistic dust as a limit case [40]) with compact support always contained in a finite world-tube W, which in the Cartesian 4-coordinates of an inertial system is a time-like cylinder of radius R. At the initial time the disk support $D_{\tau=0}$ is just the circle W_o of radius R in the chosen inertial system; at subsequent times the support could be different according to the internal dynamics of the isolated system. Let us consider a parametrized Minkowski theory [27, 88, 89], namely a Lagrangian whose configuration variables are those of the isolated system plus the embeddings $z^\mu(\tau,\vec{\sigma})$ describing the allowed 3+1 splittings of Minkowski space-time with the associated notions of Cauchy and simultaneity surfaces as said in the Introduction. Since the embeddings $z^\mu(\tau,\vec{\sigma})$ are *gauge variables* all the allowed notions of simultaneity are gauge equivalent. The simultaneity surfaces Σ_τ (in general, but not necessarily, curved Riemannian 3-manifolds embedded in the flat Minkowski space-time) intersect the world-tube W with 3-dimensional sub-manifolds $D_\tau \subset \Sigma_\tau$ describing the instantaneous 3-space of the disk at τ according to this notion of simultaneity.

Let us choose a particular embedding $z^\mu(\tau,\vec{\sigma})$, i.e. a well defined notion of simultaneity. The congruence of time-like (in general non-inertial) observers whose world-lines are the integral curves of the vector filed $l^\mu(\tau,\vec{\sigma})$ of unit normals to Σ_τ is used to define *rods and clocks* for this notion of simultaneity by slow transport of those pertaining to the asymptotic inertial observers at spatial infinity (fixed stars), which are the standard rods and clocks of inertial systems. Alternatively, we can define the radar 4-coordinates $(\tau,\vec{\sigma})$ with the method of Subsection A. Therefore on Σ_τ we can measure spatial distances with the 3-metric g_{rs}, synchronize distant clocks and define one-way velocity of light between two simultaneity surfaces as discussed in Section II.

The second congruence associated to the chosen notion of simultaneity, whose time-like observers have the integral curves of the vector field $z^\mu_\tau(\tau,\vec{\sigma})/\sqrt{\epsilon\, g_{\tau\tau}(\tau,\vec{\sigma})}$ as world-lines [41],

[40] As an example of a congruence simulating a geometrical rotating disk we can consider the relativistic dust described by generalized Eulerian coordinates of Ref.[107] after the gauge fixing to a family of differentially rotating parallel hyper-planes.

[41] They are the lines $\vec{\sigma} = \vec{\sigma}_o = const.$, i.e. the generalized helices $x^\mu_{\vec{\sigma}_o}(\tau) = z^\mu(\tau,\vec{\sigma}_o)$ with

is used to define a (in general non-surface-forming, non-synchronizable) reference frame with translational and rotational accelerations, whose restriction $u^\mu_D(\tau,\vec{\sigma})$ to the world-tube W [42] is the 3+1 counterpart of the local rigid rotating reference frame used in the treatments of the rotating disk (see for instance Ref.[108]).

Every notion of simultaneity has associated a different notion of spatial length, and therefore a different radius and circumference length will appear at each non-inertial observers, namely the disk 3-geometry will be simultaneity-dependent. But this is natural because in special relativity the notions of simultaneity and simultaneous spatial distance are reference-frame-dependent i.e. observer-dependent. Even if all of them are gauge-equivalent in parametrized Minkowski theories, there is no useful notion of *gauge equivalence class* (see Ref. [8] for the analogous problem in general relativity), because an extended physical laboratory corresponds to a completely fixed gauge and not to an equivalence class: its definition requires a definite choice of the notion of simultaneity and of a reference observer, endowed with a tetrad, as origin of the coordinates.

$\dot{x}^\mu_{\vec{\sigma}_o}(\tau)/\sqrt{\epsilon\, \dot{x}^2_{\vec{\sigma}_o}(\tau)} = z^\mu_\tau(\tau,\vec{\sigma}_o)/\sqrt{\epsilon\, g_{\tau\tau}(\tau,\vec{\sigma}_o)}$.

[42] In general for given a world-tube W there will be a preferred family of adapted embeddings such that the associated vector fields $u^\mu_D(\tau,\vec{\sigma})$ have the property that their integral lines are contained completely in the world-tube W. In the next Subsection we will study a simple embedding of the type (4.1) with this property.

C. The Simplest Embedding for a Rotating Disk and the Sagnac Effect.

Let us now consider the following admissible embedding of the type (4.1), corresponding to a foliation with flat parallel space-like hyper-planes with normal l^μ (defining inertial Eulerian observers)

$$z^\mu(\tau, \vec{\sigma}) = l^\mu\, \tau + \epsilon^\mu_r\, R^r_{(3)\,s}(\tau, \sigma)\, \sigma^s, \tag{6.17}$$

where

$$\begin{aligned}
R^r_{(3)\,s}(\tau, \sigma) &= \begin{pmatrix} \cos\,\theta(\tau,\sigma) & -\sin\,\theta(\tau,\sigma) & 0 \\ \sin\,\theta(\tau,\sigma) & \cos\,\theta(\tau,\sigma) & 0 \\ 0 & 0 & 1 \end{pmatrix},\\
\theta(\tau, \sigma) &= F(\sigma)\, \omega\, \tau, \quad F(\sigma) < \frac{c}{\omega\, \sigma},\\
\Omega^r{}_s(\tau, \sigma) &= \Big(R^{-1}_{(3)}\, \frac{d R_{(3)}}{d\tau}\Big)^r{}_s(\tau, \sigma) = \omega\, F(\sigma) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\\
\Omega(\tau, \sigma) &= \Omega(\sigma) = \omega\, F(\sigma).
\end{aligned} \tag{6.18}$$

A simple choice for the function $F(\sigma)$, compatible with the conditions (4.14), is $F(\sigma) = \frac{1}{1 + \frac{\omega^2\, \sigma^2}{c^2}}$ [43], so that at spatial infinity we get $\Omega(\tau, \sigma) = \frac{\omega}{1 + \frac{\omega^2\, \sigma^2}{c^2}} \rightarrow_{\sigma \rightarrow \infty} 0$. Let us remark that nearly rigid rotating systems can be obtained by using a function $F(\sigma)$ approximating the step function $\Theta(R - r)$.

By introducing cylindrical 3-coordinates r, φ, h we get the following form of the embedding

[43] We have introduced the explicit c dependence. In the rest of the Section we put $c = 1$.

$$
\begin{aligned}
z^\mu(\tau,\vec{\sigma}) &= l^\mu\,\tau + \epsilon^\mu_1\,[\cos\,\theta(\tau,\sigma)\,\sigma^1 - \sin\,\theta(\tau,\sigma)\,\sigma^2] + \\
&+\; \epsilon^\mu_2\,[\sin\,\theta(\tau,\sigma)\,\sigma^1 + \cos\,\theta(\tau,\sigma)\,\sigma^2] + \epsilon^\mu_3\,\sigma^3 = \\
&= l^\mu\,\tau + \epsilon^\mu_1\,r\,\cos\,[\theta(\tau,\sigma)+\varphi] + \epsilon^\mu_2\,r\,\sin\,[\theta(\tau,\sigma)+\varphi] + \epsilon^\mu_3\,h,\\
&\sigma^1 = r\,\cos\,\varphi,\quad \sigma^2 = r\,\sin\,\varphi,\quad \sigma^3 = h,\quad \sigma = \sqrt{r^2+h^2}.
\end{aligned}
\tag{6.19}
$$

Then we get

$$
\begin{aligned}
\frac{\partial z^\mu(\tau,\vec{\sigma})}{\partial\tau} &= z^\mu_\tau(\tau,\vec{\sigma}) = l^\mu - \omega\,r\,F(\sigma)\,\Big(\epsilon^\mu_1\,\sin\,[\theta(\tau,\sigma)+\varphi] - \epsilon^\mu_2\,\cos\,[\theta(\tau,\sigma)+\varphi]\Big),\\
\frac{\partial z^\mu(\tau,\vec{\sigma})}{\partial\varphi} &= z^\mu_\varphi(\tau,\vec{\sigma}) = -\epsilon^\mu_1\,r\,\sin\,[\theta(\tau,\sigma)+\varphi] + \epsilon^\mu_2\,r\,\cos\,[\theta(\tau,\sigma)+\varphi]\\
\frac{\partial z^\mu(\tau,\vec{\sigma})}{\partial r} &= z^\mu_{(r)}(\tau,\vec{\sigma}) = -\epsilon^\mu_1\,\Bigg((\cos\,[\theta(\tau,\sigma)+\varphi] - \frac{r^2\omega\tau}{\sqrt{r^2+h^2}}\,\frac{dF(\sigma)}{d\sigma}\,\sin\,[\theta(\tau,\sigma)+\varphi]\Bigg) + \\
&+\; \epsilon^\mu_2\,\Bigg(\sin\,[\theta(\tau,\sigma)+\varphi] + \frac{r^2\omega\tau}{\sqrt{r^2+h^2}}\,\cos\,[\theta(\tau,\sigma)+\varphi]\Bigg)\\
\frac{\partial z^\mu(\tau,\vec{\sigma})}{\partial h} &= z^\mu_h(\tau,\vec{\sigma}) = \epsilon^\mu_3 - \epsilon^\mu_1\,\Bigg(\frac{rh\omega\tau}{\sqrt{r^2+h^2}}\,\frac{dF(\sigma)}{d\sigma}\,\sin\,[\theta(\tau,\sigma)+\varphi]\Bigg) + \\
&+\; \epsilon^\mu_2\,\Bigg(\frac{rh\omega\tau}{\sqrt{r^2+h^2}}\,\frac{dF(\sigma)}{d\sigma}\,\cos\,[\theta(\tau,\sigma)+\varphi]\Bigg),
\end{aligned}
\tag{6.20}
$$

where we have used the notation (r) to avoid confusion with the index r used as 3-vector index (for example in σ^r).

Then in cylindrical 4-coordinates τ, r, φ and h the 4-metric is

$$
\begin{aligned}
\epsilon\, g_{\tau\tau}(\tau,\vec{\sigma}) &= 1-\omega^2\, r^2\, F^2(\sigma),\\
\epsilon\, g_{\tau\varphi}(\tau,\vec{\sigma}) &= -\omega\, r^2\, F(\sigma),\\
\epsilon\, g_{\tau(r)}(\tau,\vec{\sigma}) &= -\frac{\omega^2\, r^3\, \tau}{\sqrt{r^2+h^2}}\, F(\sigma)\, \frac{dF(\sigma)}{d\sigma},\\
\epsilon\, g_{\tau h}(\tau,\vec{\sigma}) &= -\frac{\omega^2\, r^2\, h\, \tau}{\sqrt{r^2+h^2}}\, F(\sigma)\, \frac{dF(\sigma)}{d\sigma},\\
\epsilon\, g_{\varphi\varphi}(\tau,\vec{\sigma}) &= -r^2,\\
\epsilon\, g_{(r)(r)}(\tau,\vec{\sigma}) &= -1-\frac{r^4\, \omega^2\, \tau^2}{r^2+h^2}\left(\frac{dF(\sigma)}{d\sigma}\right)^2,\\
\epsilon\, g_{hh}(\tau,\vec{\sigma}) &= -1-\frac{r^2\, h^2\, \omega^2\, \tau^2}{r^2+h^2}\left(\frac{dF(\sigma)}{d\sigma}\right)^2,\\
\epsilon\, g_{(r)\varphi}(\tau,\vec{\sigma}) &= -\frac{\omega\, r^3\, \tau}{\sqrt{r^2+h^2}}\, \frac{dF(\sigma)}{d\sigma},\\
\epsilon\, g_{h\varphi}(\tau,\vec{\sigma}) &= -\frac{\omega^2\, r^2\, h\, \tau}{\sqrt{r^2+h^2}}\, \frac{dF(\sigma)}{d\sigma},\\
\epsilon\, g_{h(r)}(\tau,\vec{\sigma}) &= -\frac{r^3\, h\, \omega^2\, \tau^2}{r^2+h^2}\left(\frac{dF(\sigma)}{d\sigma}\right)^2,\\
& \quad with\, inverse\\
\epsilon\, g^{\tau\tau}(\tau,\vec{\sigma}) &= 1, \qquad \epsilon\, g^{\tau\varphi}(\tau,\vec{\sigma}) = -\omega\, F(\sigma),\\
\epsilon\, g^{\tau(r)}(\tau,\vec{\sigma}) &= \epsilon\, g^{\tau h}(\tau,\vec{\sigma}) = 0, \qquad \epsilon\, g^{(r)(r)}(\tau,\vec{\sigma}) = \epsilon\, g^{hh}(\tau,\vec{\sigma}) = -1,\\
\epsilon\, g^{\varphi\varphi}(\tau,\vec{\sigma}) &= -\frac{1+\omega^2\, r^2\, [\tau^2\, (\frac{dF(\sigma)}{d\sigma})^2 - F^2(\sigma)}{r^2},\\
\epsilon\, g^{\varphi(r)}(\tau,\vec{\sigma}) &= \frac{\omega\, r\, \tau}{\sqrt{r^2+h^2}}\, \frac{dF(\sigma)}{d\sigma}, \qquad \epsilon\, g^{\varphi h}(\tau,\vec{\sigma}) = \frac{\omega\, h\, \tau}{\sqrt{r^2+h^2}}\, \frac{dF(\sigma)}{d\sigma}.
\end{aligned}
\tag{6.21}
$$

It is easy to observe that the congruence of (non inertial) observers defined by the 4-velocity field

$$\frac{z^\mu_\tau(\tau,\vec{\sigma})}{\sqrt{\epsilon\, g_{\tau\tau}(\tau,\vec{\sigma})}} = \frac{l^\mu - \omega\, r\, F(\sigma)\, \Big(\epsilon^\mu_1\, \sin\left[\theta(\tau,\sigma) + \varphi\right] - \epsilon^\mu_2\, \cos\left[\theta(\tau,\sigma)+\varphi\right]\Big)}{1 - \omega^2\, r^2\, F^2(\sigma)}, \tag{6.22}$$

has the observers moving along the world-lines

$$\begin{aligned} x^\mu_{\vec{\sigma}_o}(\tau) &= z^\mu(\tau,\vec{\sigma}_o) = \\ &= l^\mu\, \tau + r_o\, \Big(\epsilon^\mu_1\, \cos\left[\omega\,\tau\, F(\sigma_o) + \varphi_o\right] + \epsilon^\mu_2\, \sin\left[\omega\,\tau\, F(\sigma_o) + \varphi_o\right]\Big) + \epsilon^\mu_3\, h_o. \end{aligned} \tag{6.23}$$

The world-lines (6.23) are labeled by their initial value $\vec{\sigma} = \vec{\sigma}_o = (\varphi_o, r_o, h_o)$ at $\tau = 0$.

In particular for $h_o = 0$ and $r_o = R$ these world-lines are helices on the *cylinder* in the Minkowski space

$$\begin{aligned} &\epsilon^\mu_3\, z_\mu = 0, \qquad (\epsilon^\mu_1\, z_\mu)^2 + (\epsilon^\mu_2\, z_\mu)^2 = R^2, \\ &or \qquad r = R, \qquad h = 0. \end{aligned} \tag{6.24}$$

These helices are defined the equations $\varphi = \varphi_o$, $r = R$, $h = 0$ if expressed in the embedding adapted coordinates φ, r, h. Then the congruence of observers (6.22), defined by the foliation (6.17), defines on the cylinder (6.24) the *rotating observers* usually assigned to the rim of a rotating disk, namely observes running along the helices $x^\mu_{\vec{\sigma}_o}(\tau) = l^\mu\, \tau + R\, \Big(\epsilon^\mu_1\, \cos\left[\Omega(R)\, \tau\, + \varphi_o\right] + \epsilon^\mu_2\, \sin\left[\Omega(R)\,\tau\, + \varphi_o\right]\Big)$ after having put $\Omega(R) \equiv \omega\, F(R)$.

On the cylinder (6.24) the line element is obtained from the line element ds^2 for the metric (6.21) by putting $dh = dr = 0$ and $r = R$, $h = 0$. Therefore the cylinder line element is

$$\epsilon\, (ds_{cyl})^2 = \Big[1 - \omega^2\, R^2\, F^2(R)\Big]\, (d\tau)^2 - 2\,\omega\, R^2\, F(R)\, d\tau d\varphi - R^2\, (d\varphi)^2. \tag{6.25}$$

We can define the light rays on the cylinder, i.e. the null curves on it, by solving the equation

$$\boldsymbol{\epsilon}\,(ds_{cyl})^2 = (1 - R^2\,\Omega^2(R))\,d\tau^2 - 2\,R^2\,\Omega(R)\,d\tau\,d\varphi - R^2\,d\varphi^2 = 0, \tag{6.26}$$

which implies

$$R^2\left(\frac{d\varphi(\tau)}{d\tau}\right)^2 + 2\,R^2\,\Omega(R)\left(\frac{d\varphi(\tau)}{d\tau}\right) - (1 - R^2\,\Omega(R)) = 0. \tag{6.27}$$

The two solutions

$$\frac{d\varphi(\tau)}{d\tau} = \pm\,\frac{1}{R} - \Omega(R), \tag{6.28}$$

define the world-lines on the cylinder for *clockwise or anti-clockwise* rays of light.

$$\begin{array}{ll} \Gamma_1: & \varphi(\tau) - \varphi_o = \left(+\frac{1}{R} - \Omega(R)\right)\,\tau, \\ \Gamma_2: & \varphi(\tau) - \varphi_o = \left(-\frac{1}{R} - \Omega(R)\right)\,\tau \end{array}. \tag{6.29}$$

This is the *geometric origin* of the *Sagnac Effect*. Since Γ_1 describes the world-line of the ray of light emitted at $\tau = 0$ by the rotating observer $\varphi = \varphi_o$ in the increasing sense of φ (anti-clockwise), while Γ_2 describes that of the ray of light emitted at $\tau = 0$ by the same observer in the decreasing sense of φ (clockwise), then the two rays of light will be re-absorbed by the same observer at *different τ-times* [44] $\tau_{(\pm\,2\pi)}$, whose value, determined by the two conditions $\varphi(\tau_{(\pm\,2\pi)}) - \varphi_o = \pm\,2\pi$, is

$$\begin{array}{ll} \Gamma_1: & \tau_{(+2\pi)} = \frac{2\pi\,R}{1-\Omega(R)\,R}, \\ \Gamma_2: & \tau_{(-2\pi)} = \frac{2\pi\,R}{1+\Omega(R)\,R}. \end{array} \tag{6.30}$$

The *time difference* between the re-absorption of the two light rays is

$$\Delta\tau = \tau_{(+2\pi)} - \tau_{(-2\pi)} = \frac{4\pi\,R^2\,\Omega(R)}{1 - \Omega^2(R)\,R^2} = \frac{4\pi\,R^2\,\omega\,F(R)}{1 - \omega^2\,F^2(R)\,R^2}, \tag{6.31}$$

and it corresponds to the phase difference named the *Sagnac effect*

[44] Sometimes the *proper time of the rotating observer* is used: $d\mathcal{T}_o = d\tau\sqrt{1 - \Omega^2(R)\,R^2}$.

$$\Delta\Phi = \Omega\,\Delta\tau. \tag{6.32}$$

We see that we recover the standard result if we take a function $F(\sigma)$ such that $F(R) = 1$. In the non-relativistic applications, where $F(\sigma) \rightarrow 1$, the correction implied by admissible relativistic coordinates is totally irrelevant.

Till now we have described the Sagnac effect by using the τ time coordinate associated to the notion of simultaneity (4.1). Let us now compare it with the notions of synchronization for the rotating observers based on the use of the world-lines (6.29) for the two light rays. This will be done by using the notions of synchronizations of type A) and B) introduced in Section II. Then we will study the associated notions of spatial distance of type A) or B) by evaluating the radius and the circumference of the rotating disk in the two cases.

Let us consider a reference observer $(\varphi_o = const., \tau)$ and another one $(\varphi = const. \neq \varphi_o, \tau)$. If $\varphi > \varphi_o$ we use the notation (φ_R, τ), while for $\varphi < \varphi_o$ the notation (φ_L, τ) with $\varphi_R - \varphi_o = -(\varphi_L - \varphi_o)$.

Let us consider the two rays of light Γ_{R-} and Γ_{L-}, emitted in the right and left directions at the event (φ_o, τ_-) on the rim of the disk and received at τ at the events (φ_R, τ) and (φ_L, τ) respectively. Both of them are reflected towards the reference observer, so that we have two rays of light Γ_{R+} and Γ_{L+} which will be absorbed at the event (φ_o, τ_+).

By using Eq.(6.30) for the light propagation, we get

$$\begin{aligned}
\Gamma_{R-}: &\qquad (\varphi - \varphi_o) = \tfrac{1 - R\Omega(R)}{R}\,(\tau - \tau_-),\\
\Gamma_{R+}: &\qquad (\varphi - \varphi_o) = \tfrac{1 + R\Omega(R)}{R}\,(\tau_+ - \tau),\\
\Gamma_{L-}: &\qquad (\varphi - \varphi_o) = -\tfrac{1 + R\Omega(R)}{R}\,(\tau - \tau_-),\\
\Gamma_{L+}: &\qquad (\varphi - \varphi_o) = -\tfrac{1 - R\Omega(R)}{R}\,(\tau_+ - \tau).
\end{aligned} \tag{6.33}$$

Eqs.(2.17),(2.18) define the following local synchronization of type B) in a neighborhood of the observer (φ_o, τ) [(φ, τ) is an observer in the neighborhood]

$$c\,\Delta\,\widetilde{\mathcal{T}} = \sqrt{1 - R^2\,\Omega^2(R)}\,\Delta\tau_E = \sqrt{1 - R^2\,\Omega^2(R)}\,\Delta\tau - \frac{R^2\,\Omega^2(R)}{\sqrt{1 - R^2\Omega^2(R)}}\,\Delta\varphi. \tag{6.34}$$

Let us see what happens if we try to extend this local synchronization of type B) to a global one for two distant observers (φ_o, τ) and (φ, τ) in the form of an Einstein convention for τ_E (the result is the same both for $\varphi = \varphi_R$ and $\varphi = \varphi_L$)

$$\tau_E = \frac{1}{2}(\tau_+ + \tau_-) = \tau - \frac{R^2 \Omega(R)}{1 - R^2\, \Omega^2(R)} (\varphi - \varphi_o). \tag{6.35}$$

This is contradictory because the curves defined by $\tau_E = constant$ are *not closed*, since they are helices that assign the *same time* τ_E to different events on the world-line of an observer $\varphi_o = constant$. For example

$$(\varphi_o, \tau) \quad \text{and} \quad \Big(\varphi_o, \tau + 2\pi \frac{R^2 \Omega(R)}{1 - R^2\, \Omega^2(R)}\Big)$$

are on the same helix $\tau_E = constant$.

This *desynchronization effect or synchronization gap* is only the expression of the fact that the observers of the rotating disk congruence with non-zero vorticity are not globally synchronizable, i.e. that the B) synchronization holds only locally in the form (6.34) [45]. As a consequence *usually a discontinuity in the synchronization of clocks is accepted and taken into account* (see Ref.[60] for the GPS). *Instead, with an admissible notion of simultaneity, all the clocks on the rim of the rotating disk lying on a hyper-surface Σ_τ are automatically synchronized.*

The synchronization of the type A) is defined by the condition $\tau = const.$ and can be built with a generalized operative procedure as discussed in Subsection A. In fact by Eqs.(6.33) we can calculate τ and φ as function on $\tau_\pm$ e $n = \pm$ [$n = +$ for $\varphi = \varphi_R$, $n = -$ for $\varphi = \varphi_L$; n replace $\hat{n}_{(\tau_-)}(\theta_{(\tau_-)}, \phi_{(\tau_-)})$ of Eq.(6.11)] and obtain the following modification of Einstein's convention for radar time

$$\tau(\tau_-, n, \tau_+) \;=\; \frac{1}{2}(\tau_+ + \tau_-) - \frac{n\, R\, \Omega(R)}{2}\, (\tau_+ - \tau_-) \stackrel{def}{=} \tau_- + \mathcal{E}(\tau_-, n, \tau_+)\, (\tau_+ - \tau_-),$$

$$with \qquad \mathcal{E}(\tau_-, n, \tau_+) = \frac{1 - n\, R\, \Omega(R)}{2}, \qquad \Omega(R) = \omega\, F(R). \tag{6.36}$$

[45] See Ref.[59] for a derivation of the Sagnac effect in an inertial system by using Einstein's synchronization in the locally comoving inertial frames on the rim of the disk and by asking for the equality of the one-way velocities in opposite directions.

Let us now consider the evaluation of the radius and the circumference of the rotating disk with the A) and B) notions of spatial distance.

With the convention A) on the hyper-surfaces (6.17) we use the 3-metric $-\boldsymbol{\epsilon}\, g_{su}\,(\tau, u = (r), \varphi, h)$ given by Eq.(6.21). With this metric the length of the circumference

$$\epsilon^{\mu}_3\, z_\mu = 0, \quad (\epsilon^{\mu}_1\, z_\mu)^2 + (\epsilon^{\mu}_2\, z_\mu)^2 = R, \qquad or \qquad h = 0, \quad r = R, \tag{6.37}$$

is

$$C = \int_0^{2\pi} d\varphi\, \sqrt{-\boldsymbol{\epsilon}\, g_{\varphi\varphi}} = 2\pi\, R, \tag{6.38}$$

The curve

$$h = 0, \qquad \varphi = \varphi_o = constant, \tag{6.39}$$

has the length

$$\widetilde{R} = \int_0^R dr\, \sqrt{-\boldsymbol{\epsilon}\, g_{(r)(r)}} = \int_0^R dr\, \sqrt{1 - r^2\, \omega^2\, \tau^2\, \Big(\frac{dF(r)}{dr}\Big)^2}, \tag{6.40}$$

which is equal to R only at $\tau = 0$.

However the curve (6.39) does not decribe a ray of the circumference (6.37). The curve (6.39) has the parametric representation with parameter r

$$\epsilon^{\mu}_3\, z_\mu = 0, \qquad \epsilon^{\mu}_1\, z_\mu = r\cos[\varphi_o + \omega\, \tau\, F(r)], \qquad \epsilon^{\mu}_2\, z_\mu = r\sin[\varphi_o + \omega\, \tau\, F(r)], \tag{6.41}$$

while a ray of the circumference (6.37) has the parametric representation

$$
\begin{aligned}
\epsilon^{\mu}_{3}\, z_{\mu} &= 0,\\
\epsilon^{\mu}_{1}\, z_{\mu} &= r\cos\varphi_o = r\cos[\varphi(\tau,r) + \omega\,\tau\, F(r)],\\
\epsilon^{\mu}_{2}\, z_{\mu} &= r\sin\varphi_o = r\sin[\varphi(\tau,r) + \omega\,\tau\, F(r)],\\
&\mathit{or} \qquad h = 0, \qquad r = \lambda, \qquad \varphi(\tau) = \varphi_o - \omega\,\tau\, F(\lambda).
\end{aligned}
\tag{6.42}
$$

The length of the true ray (6.42) is [remarkably this result is independent of the function $F(r)$]

$$
\int_0^R d\lambda \sqrt{-\boldsymbol{\epsilon}\left[g_{(r)(r)}\left(\frac{dr}{d\lambda}\right)^2 + 2g_{(r)\varphi}\left(\frac{dr}{d\lambda}\right)\left(\frac{d\varphi}{d\lambda}\right) + g_{\varphi\varphi}\left(\frac{d\varphi}{d\lambda}\right)^2\right]} = R. \tag{6.43}
$$

Moreover let us observe that the 3-dimensional tensor of curvature obtained by the 3-metric g_{rs} is null, ${}^3R_{suvw} = 0$, since the rotating coordinates r, φ, h can be obtained by a τ-dependent coordinate transformation by the Cartesian 3-coordinate on the (flat) hyper-plane (6.17): $\zeta^r = \epsilon^r_{\mu}\, z^{\mu}$. Therefore, with the admissible notion of simultaneity (6.17) the 3-geometry of every slice of the rotating disk contained in the simultaneity surfaces Σ_τ is Euclidean.

On the contrary with the convention B), used by most authors but without the function $F(r)$ ensuring an admissible foliation, we have to use the 3-metric

$$
{}^3\gamma_{uv} = -\boldsymbol{\epsilon}\left(g_{uv} - \frac{g_{\tau u}\, g_{\tau v}}{g_{\tau\tau}}\right). \tag{6.44}
$$

Since on the plane $h = 0$ we get (note the φ-independence and also the τ-independence of ${}^3\gamma_{\varphi\varphi}\Big|_{h=0}$)

$$
\begin{aligned}
{}^3\gamma_{\varphi\varphi}\Big|_{h=0} &= \boldsymbol{\epsilon}\,\left[-g_{\varphi\varphi} + \frac{g_{\tau\varphi}\, g_{\tau\varphi}}{g_{\tau\tau}}\right]_{h=0} = \frac{r^2}{1-r^2\Omega^2(r)},\\
{}^3\gamma_{(r)(r)}\Big|_{h=0} &= \boldsymbol{\epsilon}\,\left[-g_{(r)(r)} + \frac{g_{\tau(r)}\, g_{\tau(r)}}{g_{\tau\tau}}\right]_{h=0} = 1 + \frac{r^2\omega^2\tau^2}{1-r^2\Omega^2(r)}\left(\frac{dF(r)}{dr}\right)^2,\\
{}^3\gamma_{\varphi(r)}\Big|_{h=0} &= \boldsymbol{\epsilon}\,\left[-g_{\varphi(r)} + \frac{g_{\tau\varphi}\, g_{\tau(r)}}{g_{\tau\tau}}\right]_{h=0} = \frac{\omega\tau r^2}{1-r^2\Omega^2(r)}\frac{dF(r)}{dr},
\end{aligned}
\tag{6.45}
$$

we obtain the following length for the circumference (6.37)

$$
C' = \int_0^{2\pi} d\varphi\, \sqrt{{}^3\gamma_{\varphi\varphi}} = \frac{2\pi\, R}{\sqrt{1-R^2\Omega^2(R)}},
\tag{6.46}
$$

while the length of the ray (6.42) is [this result is independent from $F(r)$]

$$
\int_0^R d\lambda \sqrt{{}^3\gamma_{(r)(r)}\left(\frac{dr}{d\lambda}\right)^2 + 2\,{}^3\gamma_{(r)\varphi}\left(\frac{dr}{d\lambda}\right)\left(\frac{d\varphi}{d\lambda}\right) + {}^3\gamma_{\varphi\varphi}\left(\frac{d\varphi}{d\lambda}\right)^2} = R.
\tag{6.47}
$$

The metric ${}^3\gamma_{uv}$ defines a curvature tensor ${}_\gamma R_{suvw} \neq 0$ (see Ref.[55]). Therefore, a non-Euclidean 3-geometry for the rotating disk is obtained if we approximate the instantaneous 3-space of the disk with anyone of the local rest frames of the observers of the congruence with non-zero vorticity (the abstract relative space of Ref.[55]) on the rim of the disk (Eq.(6.45) is φ-independent).

D. Relativistic Theory for Time and Frequency Transfer.

As a further application of the admissible foliations let us consider the problem of the evaluation of the time and frequency transfers [80] from an Earth station B and a satellite A, because it is relevant for the ACES ESA project on synchronization of clocks [77], which needs corrections of order c^{-3} due to the level of accuracy in time keeping ($5 \cdot 10^{-17}$ in fractional frequency or $5\,ps$ in time transfer) reached with laser cooled atomic clocks.

As we shall see the ACES mission can be re-interpreted as a determination of the function $\mathcal{E}$ of Eq.(6.12), measuring the deviation from Einstein's convention ($\mathcal{E} = \frac{1}{2}$), which is associated to a choice of the notion of simultaneity compatible with admissible differentially rotating 4-coordinates taking into account the rotation of the Earth. Since, as we have seen, such a choice is one of the conventions defining an enlarged laboratory in special relativity, it has to be done *a priori* and in the most convenient way.

The existing calculation of these quantities [60, 80] has been done in the non-inertial (non-rotating) Geocentric Celestial Reference Frame [3] (see footnote 1) considered as an inertial frame in free fall in post-Newtonian gravity [46] and uses the hyper-planes of constant geocentric coordinate time as notion of simultaneity. For attempts to re-formulate the problem in non-inertial frames, taking into account the rotation of the Earth, see Refs.[81], especially the first one where for the first time there is an application of the PPN formalism to the time transfer problem with estimates of the effects of Earth multipoles to the ACES project.

Let us first review the approach of Ref.[80]. If x^μ are Cartesian inertial Geocentric 4-coordinates, with the notion of simultaneity based on the hyper-planes $x^o = c\,t = const.$ (Einstein's convention), the world-line of the Earth station B is parametrized as $x^\mu_B(t) = (x^o_B = c\,t; \vec{x}_B(t)\,)$, while the world-line of the satellite A is $x^\mu_A(t) = (x^o_A = c\,t; \vec{x}_A(t)\,)$. The basic quantity to be evaluated with the simultaneity $x^o = const.$ is the *one-way time transfer*.

[46] The line element is modified to take into account post-Newtonian gravitational effects in a suitable harmonic 4-coordinate system *identified* with an inertial geocentric Cartesian (non-rotating) coordinate system. Post-Newtonian gravity is needed for the evaluation [80] of photon world-lines and Shapiro time delay. Strictly speaking, given the post-Newtonian 4-metric, Eistein's convention is not compatible with it and one should replace the inertial system with an admissible (non-Cartesian) radar 4-coordinate system generating it, in analogy with Eq.(3.7) for Møller rotating 4-metric. This radar 4-coordinate system, and its notion of simultaneity, should then be modified to take into account Earth rotation along the lines presented in this Subsection.

If at $t = t_A$ the satellite A emits an electro-magnetic signal, its reception at the Earth station B will happen at time $t_B > t_A$ such that $\triangle^2_{AB} = (x_A - x_B)^2 = \boldsymbol{\epsilon}\, [c^2\, (t_A - t_B)^2 - \vec{\triangle}^2_{AB}] = 0$ with $\vec{\triangle}_{AB} = \vec{x}_A(t_A) - \vec{x}_B(t_B) \stackrel{def}{=} R_{AB}\, \hat{N}_{AB}$, $\hat{N}^2_{AB} = 1$ (we use a notation like in Ref.[80] for comparison). Then in the flat Minkowski space-time limit we get

$$T_{AB} = t_B - t_A = \frac{1}{c}\, R_{AB}. \tag{6.48}$$

Since in real experiments the position $\vec{x}_B(t_A)$ of the Earth station at the emission time is better known than the position $\vec{x}_B(t_B)$ at the reception time, the quantity R_{AB} has to be re-expressed in terms of the *instantaneous* (in the sense of the simultaneity $x^o = const.$) distance $\vec{D}_{AB} = \vec{x}_A(t_A) - \vec{x}_B(t_A)$, $D_{AB} = |\vec{D}_{AB}|$. To order c^{-3} we get [80]

$$\begin{aligned}
R_{AB} &= |\vec{x}_A(t_A) - \vec{x}_B(t_B)| = \sqrt{\Big[\vec{D}_{AB} + \vec{v}_B(t_A)\, R_{AB} + \frac{1}{2}\, \vec{a}_B(t_A)\, R^2_{AB} + O(R^3_{AB})\Big]^2},\\
&\Downarrow\\
R_{AB} &= D_{AB} + \frac{1}{c}\, \vec{D}_{AB} \cdot \vec{v}_B(t_A) + \\
&+ \frac{1}{c^2}\, D_{AB} \Big[\vec{v}^2_B(t_A) + \frac{(\vec{D}_{AB} \cdot \vec{v}_B(t_A)}{D^2_{AB}} + \vec{D}_{AB} \cdot \vec{a}_B(t_A)\Big] + O(\frac{1}{c^3}),\\
&\vec{v}_B(t) = \frac{d\vec{x}_B(t)}{dt}, \qquad \vec{a}_B(t) = \frac{d^2\vec{x}_B(t)}{dt^2}.
\end{aligned} \tag{6.49}$$

Finally post-Newtonian gravity contributes with the Shapiro time delay [80], so that the final result is (M is the Earth mass)

$$\begin{aligned}
T_{AB}(t_A) &= \frac{1}{c}\, R_{AB} + \frac{2\, G\, M}{c^3}\, ln\, \frac{|\vec{x}_A(t_A)| + |\vec{x}_B(t_B)| + R_{AB}}{|\vec{x}_A(t_A)| + |\vec{x}_B(t_B)| - R_{AB}} = \\
&= \frac{1}{c}\, D_{AB} + \frac{1}{c^2}\, \vec{D}_{AB} \cdot \vec{v}_B(t_A) + \frac{1}{c^3}\, D_{AB} \Big[\vec{v}^2_B(t_A) + \frac{(\vec{D}_{AB} \cdot \vec{v}_B(t_A))^2}{D^2_{AB}} + \vec{D}_{AB} \cdot \vec{a}_B(t_A)\Big] + \\
&+ \frac{2\, G\, M}{c^3}\, ln\, \frac{|\vec{x}_A(t_A)| + |\vec{x}_B(t_A)| + D_{AB}}{|\vec{x}_A(t_A)| + |\vec{x}_B(t_A)| - D_{AB}} + O(\frac{1}{c^4})
\end{aligned} \tag{6.50}$$

The two terms in T_{AB} beyond D_{AB}/c are usually referred to as the *Sagnac terms* of first ($1/c^2$) and second ($1/c^3$) order due the rotations of the Earth and the satellite (see Ref.[79]

for their derivation by using a standard non-admissible rotating frame). In the inertial system Earth rotation is simulated with a term $\omega_E^2\, |\vec{x}_B(t_A)|$ in the acceleration $\vec{a}_B(t_A)$.

In Ref.[80], after stating that the experimental uncertainty in time of ACES will be at the level of $5\, ps$, there is an estimate, at low elevation of a satellite at $400\, Km$ of altitude, of $200\, ns$ for the first order Sagnac term, of $11\, ps$ for the Shapiro time delay and of $5\, ps$ for the second order Sagnac term.

If we consider a signal emitted at $t_{B'}$ by the Earth station, reflected at t_A from the satellite and re-absorbed at t_B by the Earth station and if $T_{AB} = t_B - t_A$ and $T_{B'A} = t_A - t_{B'}$ are the two one-way time transfers, then for the two-way process we get

$$t_A = t_{B'} + E\,(t_B - t_{B'}), \quad with \quad E = \frac{1}{2}\Big(1 + T_{B'A} - T_{AB}\Big). \tag{6.51}$$

By measuring $t_{B'}$ and t_B with the atomic clock of the Earth station and by using a theoretical determination of the two one-way transfers with the simultaneity $x^o = const.$ it should be possible to check whether in post-Newtonian gravity Einstein's convention ($E = \frac{1}{2}$ and $T_{B'} = T_{AB}$) holds or is modified. However *a priori these calculations depend on the chosen notion of simultaneity* and may change going to a non-inertial frame taking into account Earth's rotation.

For the determination of the one- and two-way frequency transfer see Ref.[80].

Let us now see what happens if we consider a good notion of simultaneity, of the type (4.1), adapted to the rotation of the Earth, i.e. admissible transformations $x^\mu \mapsto \sigma^A$ from the Cartesian geocentric inertial coordinates x^μ to intrinsic (radar-type) coordinates such that the inverse transformation $\sigma^A \mapsto x^\mu$ defines the embedding

$$\begin{aligned} x^\mu \;&=\; z^\mu(\tau, \vec{\sigma}) = x^\mu_o + l^\mu\, \tau + \epsilon^\mu_r\, \zeta^r(\tau, \vec{\sigma}),\\ \zeta^r(\tau, \vec{\sigma}) &= R^r_{Es}(\tau, \sigma)\, \sigma^s, \end{aligned} \tag{6.52}$$

where $x^\mu(\tau) = x^\mu_o + l^\mu\, \tau$ is the world-line of the center of mass of the Earth (origin of the 3-coordinates $\vec{\sigma}$) and $\epsilon^\mu_A = (\epsilon^\mu_\tau = l^\mu; \epsilon^\mu_r)$ is an asymptotic tetrad determined by the fixed stars. Let us remark that with a suitable $x^\mu(\tau) = x^\mu_o + l^\mu\, f(\tau) + \epsilon^\mu_r\, g^r(\tau)$, with

$\dot{f}^2(\tau) > \sum_r \dot{g}^r(\tau)\, \dot{g}^r(\tau)$, we could describe the (non-inertial) motion of the center of the Earth with respect to the (quasi) inertial Solar System Barycentric Celestial Reference System (see footnote 1): in this case $x^\mu \mapsto \sigma^A$ would be an admissible coordinate transformation from such an inertial system to an intrinsic coordinated system adapted to both the linear acceleration and the rotational motions of the Earth.

Since the intrinsic coordinates are adapted to the motions of the Earth, the Earth station B has fixed intrinsic 3-coordinates

$$
\begin{aligned}
\eta^r_B &= R_B\, \hat{\eta}^r_B = const., \qquad \hat{\vec{\eta}}^2_B = 1,\\
x^\mu_B(\tau) &= z^\mu(\tau, \vec{\eta}_B) = x^\mu(\tau) + \epsilon^\mu_r\, \zeta^r(\tau, \vec{\eta}_B),\\
\zeta^r(\tau, \vec{\eta}_B) &= R^r_{E\,s}(\tau, R_B)\, R_B\, \hat{\eta}^s_B. \qquad (6.53)
\end{aligned}
$$

The matrix $R_E(\tau, \sigma)$ is a rotation matrix such that $R_E(\tau, R_B)$ takes into account the effects of the rotation, precession and nutation of the Earth at the position B of the Earth station through its three Euler angles. By ignoring precession and rotation, the effect of the rotation of the Earth is parametrized by means of the matrix (corresponding to a rotation around the third axis)

$$
\begin{aligned}
R_{EB}(\tau) &\stackrel{def}{=} R_E(\tau, R_B) = \begin{pmatrix} \cos\Omega_B\,\tau & -\sin\Omega_B\,\tau & 0\\ \sin\Omega_B\,\tau & \cos\Omega_B\,\tau & 0\\ 0 & 0 & 1 \end{pmatrix},\\
\Omega_B &= \omega_E\, F(R_B) = const., \qquad \theta_B(\tau) = \Omega_B\, \tau. \qquad (6.54)
\end{aligned}
$$

If we normalize the gauge function $F(\sigma)$ so that $F(R_B) = 1$, we get $\Omega_B = \omega_E$, where ω_E is the angular velocity of the Earth assumed constant. A possible choice for $F(\sigma)$, respecting Eqs.(4.14) and such that $F(R_B) = 1$, is

$$
F(\sigma) = \frac{1 + \omega_E^2\, R_B^2}{1 + \omega_E^2\, \sigma^2} \;<\; \frac{2}{1 + \omega_E^2\, \sigma^2} \;<\; \frac{1}{\omega_E\, \sigma}, \qquad (6.55)
$$

since $\omega_E\, R_B < 1$ ($c = 1$).

Since they are needed later on, we give also the velocity and the acceleration of the Earth station

$$
\begin{aligned}
\dot{x}^\mu_B(\tau) &= \dot{x}^\mu(\tau) + \epsilon^\mu_r\, \dot{\zeta}^r(\tau, \vec{\eta}_B) = \dot{x}^\mu(\tau) + \epsilon^\mu_r\, \dot{R}^r_{EB\,s}(\tau)\, R_B\, \hat{\eta}^s_B,\\
\ddot{x}^\mu_B(\tau) &= \ddot{x}^\mu(\tau) + \epsilon^\mu_r\, \ddot{\zeta}^r(\tau, \vec{\eta}_B) = \ddot{x}^\mu(\tau) + \epsilon^\mu_r\, \ddot{R}^r_{EB\,s}(\tau)\, R_B\, \hat{\eta}^s_B,\\
\dot{R}_{EB}(\tau) &= \Omega_B \begin{pmatrix} -\sin\Omega_B\,\tau & -\cos\Omega_B\,\tau & 0\\ \cos\Omega_B\,\tau & -\sin\Omega_B\,\tau & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad \ddot{R}_{EB}(\tau) = \Omega_B^2 \begin{pmatrix} -\cos\Omega_B\,\tau & \sin\Omega_B\,\tau & 0\\ -\sin\Omega_B\,\tau & -\cos\Omega_B\,\tau & 0\\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{6.56}
$$

For inertial motion of the Earth, $x^\mu(\tau) = x^\mu_o + l^\mu\,\tau$, we have $\dot{x}^\mu(\tau) = l^\mu$ and $\ddot{x}^\mu(\tau) = 0$.

The adapted intrinsic 3-coordinates of the satellite A are $\vec{\eta}_A(\tau) = R_A(\tau)\,\hat{\eta}_A(\tau)$, $\hat{\eta}^2_A(\tau) = 1$. They are deduced from the assumed known satellite world-line parametrized with τ, i.e. $x^\mu_A(\tau) = (c\,t(\tau); \vec{x}_A(t(\tau))\,) = z^\mu(\tau, \vec{\eta}_A(\tau))$, by using Eq.(6.8). Now we have

$$
\begin{aligned}
x^\mu_A(\tau) &= z^\mu(\tau, \vec{\eta}_A(\tau)) = x^\mu(\tau) + \epsilon^\mu_r\, \zeta^r(\tau, \vec{\eta}_A(\tau)),\\
\zeta^r(\tau, \vec{\eta}_A(\tau)) &= R^r_{EAs}(\tau)\, R_A(\tau)\, \hat{\eta}^s_A(\tau),\\
R_{EA}(\tau) &= R_E(\tau, R_A(\tau)) = \begin{pmatrix} \cos\Omega_A(\tau)\,\tau & -\sin\Omega_A(\tau)\,\tau & 0\\ \sin\Omega_A(\tau)\,\tau & \cos\Omega_A(\tau)\,\tau & 0\\ 0 & 0 & 1 \end{pmatrix},\\
\Omega_A(\tau) &= \omega_E\, F(R_A(\tau)), \qquad \theta_A(\tau) = \Omega_A(\tau)\,\tau.
\end{aligned}
\tag{6.57}
$$

If we put $F(R_A(\tau)) = 1 + G(R_A(\tau))$, we get for the angular velocity of the satellite

$$
\Omega_A(\tau) = \omega_E + \omega_A(\tau), \qquad \omega_A(\tau) = \omega_E\, G(R_A(\tau)),
\tag{6.58}
$$

and Eq.(6.55) implies $\omega_A(\tau) = \omega_E\, \frac{\omega_E^2\,[R_B^2 - R_A^2(\tau)]}{1 + \omega_E^2\, R_A^2(\tau)}$.

With this notion of simultaneity we have $\triangle^\mu_{AB} = x^\mu(\tau_B) - x^\mu(\tau_A) + \epsilon^\mu_r\,[\zeta^r(\tau_B, \vec{\eta}_B) - \zeta^r(\tau_A, \vec{\eta}_A(\tau_A)]$ when $x^\mu(\tau_B) - x^\mu(\tau_A) = l^\mu\,(\tau_B - \tau_A)$ and $\triangle_{AB} = 0$ implies

$$\begin{aligned}
\mathcal{T}_{AB} &= \frac{1}{c}\,(\tau_B - \tau_A) = \frac{1}{c}\,\mathcal{R}_{AB},\\
\mathcal{R}_{AB} &= \mathcal{R}_{AB}(\tau_A, \tau_B) = |\vec{\zeta}(\tau_B, \vec{\eta}_B) - \vec{\zeta}(\tau_A, \vec{\eta}_A(\tau_A)|.
\end{aligned} \tag{6.59}$$

If the Earth follows an non-inertial world-line $x^\mu(\tau)$ in the Solar System Barycentric Celestial Reference Frame, $\triangle^2_{AB} = 0$ implies $f(\tau_B) - f(\tau_A) = |\vec{g}(\tau_B) - \vec{g}(\tau_A) + \vec{\zeta}(\tau_B, \vec{\eta}_B) - \vec{\zeta}(\tau_A, \vec{\eta}_A(\tau_A)|$ and the discussion is much more complicated and in general can be done only numerically.

To find the analogue of Eq.(6.49), we introduce the instantaneous distance $\vec{\mathcal{D}}_{AB} = \vec{\mathcal{D}}_{AB}(\tau_A) = \vec{\zeta}(\tau_A, \vec{\eta}_B) - \vec{\zeta}(\tau_A, \vec{\eta}_A(\tau_A)$ with $\mathcal{D}_{AB} = |\vec{\mathcal{D}}_{AB}| = |\vec{\zeta}(\tau_A, \vec{\eta}_B) - \vec{\zeta}(\tau_A, \vec{\eta}_A(\tau_A)|$ and we make a Taylor expansion

$$\begin{aligned}
\vec{\zeta}(\tau_B, \vec{\eta}_B) &= \vec{\zeta}(\tau_A, \vec{\zeta}_B) + \dot{\vec{\zeta}}(\tau_A, \vec{\eta}_B)\, c\, \mathcal{T}_{AB} + \frac{1}{2}\, \ddot{\vec{\zeta}}(\tau_A, \vec{\eta}_B)\, c^2\, \mathcal{T}^2_{AB} + O(c^3\, \mathcal{T}^3_{AB}),\\
&\qquad \dot{\zeta}^r(\tau_A, \vec{\eta}_B) \stackrel{def}{=} \frac{\partial \vec{\zeta}(\tau, \vec{\eta}_B)}{\partial \tau}|_{\tau=\tau_A} = \frac{\partial R^r_{Es}(\tau, R_B)}{\partial \tau}{}_{\tau=\tau_A}\; R_B\, \hat{\eta}^s_B,\\
&\qquad \ddot{\zeta}^r(\tau_A, \vec{\eta}_B) \stackrel{def}{=} \frac{\partial^2 \vec{\zeta}(\tau, \vec{\eta}_B)}{\partial \tau^2}|_{\tau=\tau_A} = \frac{\partial^2 R^r_{Es}(\tau, R_B)}{\partial \tau^2}{}_{\tau=\tau_A}\; R_B\, \hat{\eta}^s_B,\\
\mathcal{R}_{AB} &= \sqrt{[\vec{\mathcal{D}}_{AB} + \dot{\vec{\zeta}}(\tau_A, \vec{\eta}_B)\, \mathcal{R}_{AB} + \frac{1}{2}\, \ddot{\vec{\zeta}}(\tau_A, \vec{\eta}_B)\, \mathcal{R}^2_{AB} + O(\mathcal{R}^3_{AB})]^2}.
\end{aligned} \tag{6.60}$$

Therefore we get ($\tau = c\, t$)

$$\begin{aligned}
\mathcal{T}_{AB}(\tau_A) &= \frac{1}{c}\, \mathcal{R}_{AB} = \frac{1}{c}\, \mathcal{D}_{AB} + \frac{1}{c}\, \vec{\mathcal{D}}_{AB} \cdot \dot{\vec{\zeta}}(\tau_A, \vec{\eta}_B) +\\
&+ \frac{1}{2\, c}\, \mathcal{D}_{AB} \Big[\dot{\vec{\zeta}}^2(\tau_A, \vec{\eta}_B) + \frac{(\vec{\mathcal{D}}_{AB} \cdot \dot{\vec{\zeta}}(\tau_A, \vec{\eta}_B))^2}{\mathcal{D}^2_{AB}} + \vec{\mathcal{D}}_{AB} \cdot \ddot{\vec{\zeta}}(\tau_A, \vec{\eta}_B) \Big] +\\
&+ \frac{2\, G\, M}{c^3}\, ln\, \frac{R_B + R_A(\tau_A) + \mathcal{D}_{AB}}{R_B + R_A(\tau_A) - \mathcal{D}_{AB}} + O(c^{-4}).
\end{aligned} \tag{6.61}$$

For $\omega_E \rightarrow 0$, Eq.(6.61) becomes Eq.(6.50).

To evaluate explicitly this expression, let us introduce the matrix

$$\mathcal{R}_{EAB}(\tau_A, \tau_B) \;=\; R^t_{EA}(\tau_A) R_{EB}(\tau_B) = \begin{pmatrix} \cos[\Omega_B\,\tau_B - \Omega_A(\tau_A)\,\tau_A] & -\sin[\Omega_B\,\tau_B - \Omega_A(\tau_A)\,\tau_A] & 0 \\ \sin[\Omega_B\,\tau_B - \Omega_A(\tau_A)\,\tau_A] & \cos[\Omega_B\,\tau_B - \Omega_A(\tau_A)\,\tau_A] & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.62}$$

which allows to get the following result

$$\mathcal{R}^2_{AB} = R^2_A(\tau_A) + R^2_B - 2\,\eta^r_A(\tau_A)\,\sigma^s_B\,[\mathcal{R}_{EAB}(\tau_B, \tau_A)]_{rs}. \tag{6.63}$$

If we introduce the cylindrical rotating coordinates

$$\begin{aligned}
\sigma^1_B \;&=\; r_B\,\cos\varphi_B, \qquad \sigma^2_B = r_B\,\sin\varphi_B, \qquad \sigma^3_B = h_B,\\
\eta^1_A(\tau) \;&=\; r_A(\tau)\,\cos\varphi_A(\tau), \qquad \eta^2_A(\tau) = r_A(\tau)\,\sin\varphi_A(\tau), \qquad \eta^3_A(\tau) = h_A(\tau),\\
\Rightarrow \qquad R_B &= \sqrt{r^2_B + h^2_B}, \qquad R_A(\tau) = \sqrt{r^2_A(\tau) + h^2_A(\tau)},
\end{aligned} \tag{6.64}$$

then Eq.(6.59) implies

$$\begin{aligned}
\mathcal{R}^2_{AB} \;&=\; r^2_A(\tau_A) + r^2_B - 2\,r_A(\tau_A)\,r_B\,\cos[\varphi_B + \Omega_B\tau_B - \varphi_A(\tau_A) - \Omega_A(\tau_A)\,\tau_A] + \\
&+\; (h_A(\tau_A) - h_B)^2.
\end{aligned} \tag{6.65}$$

If we put this expression in Eq.(6.60), then with a straightforward calculation we obtain the following form of Eq.(6.61) [for $\omega_E \to 0$ it gives Eq.(6.50) in cylindrical coordinates]

$$
\begin{aligned}
\mathcal{T}_{AB}(\tau_A) &= \frac{1}{c}\,\mathcal{R}_{AB} = \\
&= \frac{1}{c}\sqrt{r_A^2(\tau_A) + r_B^2 + [h_A(\tau_A) - h_B]^2 + 2\,r_A(\tau_A)\,r_B\,cos\,[(\Omega_B - \Omega_A)\,\tau_A + \varphi_B - \varphi_A(\tau_A)]} - \\
&- \frac{1}{c^2}\,r_A(\tau_A)\,r_B\,\Omega_B\,sin\,[(\Omega_B - \Omega_A)\,\tau_A + \varphi_B - \varphi_A(\tau_A)] + \\
&+ \frac{1}{2\,c^3}\Big(- \sqrt{r_A^2(\tau_A) + r_B^2 + [h_A(\tau_A) - h_B]^2 + 2\,r_A(\tau_A)\,r_B\,cos\,[(\Omega_B - \Omega_A)\,\tau_A + \varphi_B - \varphi_A(\tau_A)]} \\
&\quad \Big[r_A(\tau_A)\,r_B\,\Omega_B^2\,cos\,[(\Omega_B - \Omega_A)\,\tau_A + \varphi_B - \varphi_A(\tau_A)]\Big] + \\
&+ \frac{3\,r_A(\tau_A)\,r_B\,\Omega_B^2\,sin^2\,[(\Omega_B - \Omega_A)\,\tau_A + \varphi_B - \varphi_A(\tau_A)]}{\sqrt{r_A^2(\tau_A) + r_B^2 + [h_A(\tau_A) - h_B]^2 + 2\,r_A(\tau_A)\,r_B\,cos\,[(\Omega_B - \Omega_A)\,\tau_A + \varphi_B - \varphi_A(\tau_A)]}}\Big) - \\
&- \frac{2\,G\,M}{c^3}\,ln\,\frac{R_B + R_A(\tau_A) + K}{R_B + R_A(\tau_A) - K} + O(\frac{1}{c^4}). \\
K &= \sqrt{r_A^2(\tau_A) + r_B^2 + [h_A(\tau_A) - h_B]^2 + 2\,r_A(\tau_A)\,r_B\,cos\,[(\Omega_B - \Omega_A)\,\tau_A + \varphi_B - \varphi_A(\tau_A)]}.
\end{aligned}
\tag{6.66}
$$

The admissibility of the notion of simultaneity introduces an explicit dependence on the function $F(\sigma) = 1 + G(R_A(\tau_A))$ of Eq.(4.1)in the difference of the angular velocities of the Earth station and of the satellite

$$
\begin{aligned}
&\Omega_B - \Omega_A(\tau_A) = -\omega_A(\tau_A) = -\omega_E\,G(R_A(\tau_A)), \\
&\Downarrow \quad if \quad F(\sigma) = \frac{1 + \frac{\omega_E^2}{c^2}\,R_B^2}{1 + \frac{\omega_E^2}{c^2}\,\sigma^2} \rightarrow_{c\rightarrow\infty} 1 + \frac{\omega_E^2\,(R_B^2 - \sigma^2)}{c^2} + O(\frac{1}{c^4}), \\
&\Omega_B - \Omega_A(\tau_A) = \omega_E\,\frac{\frac{\omega_E^2}{c^2}\,[R_A^2(\tau_A) - R_B^2]}{1 + \frac{\omega_E^2}{c^2}\,R_A^2(\tau_A)} = \frac{\omega_E^3}{c^2}\,[R_A^2(\tau_A) - R_B^2] + O(\frac{1}{c^4}).
\end{aligned}
\tag{6.67}
$$

As a consequence with this notion of simultaneity, for $t_A = \frac{\tau_A}{c} < \frac{\varphi_B - \varphi_A(\tau_A)}{c\,[\Omega_B - \Omega_A(\tau_A)]} = \frac{c\,[\varphi_B - \varphi_A(\tau_A)]}{\omega_E^3\,[R_A^2(\tau_A) - R_B^2]} + O(\frac{1}{c^3})$ (it is an implicit restriction on τ_A) we get

$$
\begin{aligned}
\mathcal{T}_{AB}(\tau_A) &= \frac{1}{c}\,\mathcal{R}_{AB} = \\
&= \frac{1}{c}\,\sqrt{r_A^2(\tau_A) + r_B^2 + [h_A(\tau_A) - h_B]^2 + 2\,r_A(\tau_A)\,r_B\,cos\,[\varphi_B - \varphi_A(\tau_A)]} - \\
&- \frac{1}{c^2}\,r_A(\tau_A)\,r_B\,\Omega_B\,sin\,[\varphi_B - \varphi_A(\tau_A)] + \\
&+ \frac{1}{2\,c^3}\Big(- \sqrt{r_A^2(\tau_A) + r_B^2 + [h_A(\tau_A) - h_B]^2 + 2\,r_A(\tau_A)\,r_B\,cos\,[\varphi_B - \varphi_A(\tau_A)]} \\
&\Big[r_A(\tau_A)\,r_B\,\Omega_B^2\,cos\,[\varphi_B - \varphi_A(\tau_A)]\Big] + \\
&+ \frac{3\,r_A^2(\tau_A)\,r_B^2\,\Omega_B^2\,sin^2\,[\varphi_B - \varphi_A(\tau_A)]}{\sqrt{r_A^2(\tau_A) + r_B^2 + [h_A(\tau_A) - h_B]^2 + 2\,r_A(\tau_A)\,r_B\,cos\,[\varphi_B - \varphi_A(\tau_A)]}} - \\
&- \frac{\omega_E^3\,r_A(\tau_A)\,r_B\,[r_A^2(\tau_A) - r_B^2 + h_A^2(\tau_A) - h_B^2]\,sin\,[\varphi_B - \varphi_A(\tau_A)]}{\sqrt{r_A^2(\tau_A) + r_B^2 + [h_A(\tau_A) - h_B]^2 + 2\,r_A(\tau_A)\,r_B\,cos\,[\varphi_B - \varphi_A(\tau_A)]}}\ \tau_A\Big) - \\
&- \frac{2\,G\,M}{c^3}\,ln\,\frac{R_B + R_A(\tau_A) + \tilde{K}}{R_B + R_A(\tau_A) - \tilde{K}} + O(\frac{1}{c^4}), \\
\tilde{K} &= \sqrt{r_A^2(\tau_A) + r_B^2 + [h_A(\tau_A) - h_B]^2 + 2\,r_A(\tau_A)\,r_B\,cos\,[\varphi_B - \varphi_A(\tau_A)]}.
\end{aligned}
\tag{6.68}
$$

As already said the effect of the Earth rotation is contained in the second order Sagnac term. With the choice (6.67) of the admissible notion of simultaneity there is an effect of order $\tau_A\,\omega_E^3/c^3$.

If we put the values $h_A = h_B = 0$, $r_B = 6.4\,10^3\,Km$, $r_A - r_B \approx 400 Km$, $r_A(\tau_A) \approx const. = r_A = 4.1\,10^5\,Km$, $\omega_E/c = 7.3 \times 10^{-5}\,radian/s$ [see Ref.[80] and the last of Refs.[81]], so that $\Omega_B - \Omega_A(\tau_A) = \frac{\omega_E^3}{c^2}\,(r_A^2 - r_B^2) + O(\frac{1}{c^4}) = const. \stackrel{def}{=} \omega_{AB}$, we get

$$
for \qquad \tau_A < \frac{\varphi_B - \varphi_A(\tau_A)}{\omega_{AB}},
$$

$$
\begin{aligned}
\mathcal{T}_{AB}(\tau_A) \;\approx\; & \frac{r_A}{c}\Big[1 - \frac{\omega_E\, r_B\, sin\,\alpha(\tau_A)}{c} - \\
& - \frac{\omega_E^2\, r_A\, r_B}{2c^2}\Big([cos\,\alpha(\tau_A) + \omega_E\,\tau_A\, sin\,\alpha(\tau_A)] - 3\,\frac{r_B}{r_A}\, sin^2\alpha(\tau_A)\Big)\Big] - \\
& - \frac{2G\,M}{c^3}\, ln\,\frac{1 + \sqrt{1 + 2\,\frac{r_B}{r_A}\, cos\,\alpha(\tau_A) + (\frac{r_B}{r_A})^2}}{1 - \sqrt{1 + 2\,\frac{r_B}{r_A}\, cos\,\alpha(\tau_A) + (\frac{r_B}{r_A})^2}} + O(\frac{1}{c^4}),\\
& \alpha(\tau_A) = \varphi_B - \varphi_A(\tau_A),
\end{aligned}
$$

$$
for \qquad \tau_A \geq \frac{\varphi_B - \varphi_A(\tau_A)}{\omega_{AB}},
$$

$$
\begin{aligned}
\mathcal{T}_{AB}(\tau_A) \;=\; & \frac{K}{c} - \frac{\omega_E}{c^2}\, r_A\, r_B\, sin\,(\omega_{AB}\,\tau_A + \varphi_B - \varphi_A(\tau_A)) + \\
& + \frac{r_A\, r_B\, \omega_E^2}{c^3}\Big[-K\, cos\,(\omega_{AB}\,\tau_A + \varphi_B - \varphi_A(\tau_A)) + \frac{3}{K}\, sin^2(\omega_{AB}\,\tau_A + \varphi_B - \varphi_A(\tau_A))\Big] - \\
& - \frac{2G\,M}{c^3}\, ln\,\frac{r_B + r_A + K}{r_B + r_A - K} + O(\frac{1}{c^4}),\\
K \;=\; & \sqrt{r_A^2 + r_B^2 + 2\, r_A\, r_B\, cos\,(\omega_{AB}\,\tau_A + \varphi_B - \varphi_A(\tau_A))}. \qquad (6.69)
\end{aligned}
$$

The second order Sagnac term varies from $-\frac{\omega_E^2\, r_A^2\, r_B}{2c^3}$ for $\alpha(\tau_A) = 0$ (where the first order Sagnac term is very small) to $-\frac{\omega_E^2\, r_A^2\, r_B}{2c^3}\, \omega_E\, \tau_A$ (with a linear dependence on τ_A) for $\alpha(\tau_A) = \frac{\pi}{2}$; for $tg\,\alpha(\tau_A) = 1/\omega_E\, \tau_A$ it is reduced by a factor r_B/r_A.

Let us remark that the contribution of Earth rotation changes with $F(\sigma)$, i.e. with the choice of the notion of simultaneity in the admissible family (4.1) (or in more general admissible embeddings). As a consequence, to apply consistently the formalism one should

first of all to establish a grid of admissible radar 4-coordinates around the Earth with the method of Subsection A and then estimate the implied effect of the Earth rotation.

Therefore given an admissible notion of simultaneity and admissible rotating coordinates with a fixed $F(\sigma)$, by measuring τ_B and $\tau_{B'}$ and by using $\tau_A = \tau_{B'} + \mathcal{E}\,(\tau_B - \tau_{B'})$ we have to find the resulting [$F(\sigma)$-dependent] $\mathcal{E}$ from the equation

$$\mathcal{E} \stackrel{def}{=} \frac{1}{2}\Big[1 + c\,\mathcal{T}_{B'A}(\tau_A) - c\,\mathcal{T}_{AB}(\tau_A)\Big] =$$

$$= \frac{1}{2}\Big[1 + c\,\mathcal{T}_{B'A}(\tau_{B'} + \mathcal{E}\,[\tau_B - \tau_{B'}]) - c\,\mathcal{T}_{AB}(\tau_{B'} + \mathcal{E}\,[\tau_B - \tau_{B'}])\Big]. \qquad (6.70)$$

In conclusion we have first to make a convenient convention on the notion of simultaneity and a choice of $F(\sigma)$ for the rotating coordinates, then evaluate the one-way time transfer with it and finally use the ACES mission to check if the measured deviation $\mathcal{E} \neq \frac{1}{2}$ from Einstein's convention is just the one implied by the chosen $F(\sigma)$.

Admissible notions of simultaneity like that of Eq.(4.1) should be useful also for the description of the optical one-way time transfer among the three spacecrafts of LISA project [109] for the detection of gravitational waves. Since the spacecrafts follow heliocentric orbits (forming an approximate equilater triangle), in Eq.(4.1) $x^\mu(\tau)$ should be the straight world-line of the Sun in the Solar System Barycentric Celestial Reference Frame. Since the main problem of the LISA time delay interferometry is the elimination of the laser phase noise, introduced by the Doppler tracking scheme used to track the spacecrafts with laser beams, and since the actual rotating and flexing configuration produced by the spacecraft orbits makes this task difficult, it is worthwhile to investigate whether the presence of the arbitrary function $F(\sigma)$ of Eq.(4.1) in the one-way time transfers could help in the reduction of noise. If there would be a reduction for special forms of $F(\sigma)$, i.e. for special admissible notions of simultaneity, then one could try to implement such notions by establishing a grid of suitable radar 4-coordinates like in Subsection A.

Finally Eq.(4.1) should also be instrumental for very long baseline interferometric (VLBI) experiments, reviewed in Ref.[110].

E. Maxwell Equations in Non-Inertial Reference Frames.

The description of the electro-magnetic field as a parametrized Minkowski theory has been given in Ref.[89] (see also the Appendix of Ref.[88]). The configuration variables are the admissible embeddings $z^\mu(\tau,\vec{\sigma})$ (tending to space-like hyper-planes at spatial infinity) and the Lorentz-scalar electro-magnetic potential $A_A(\tau,\vec{\sigma}) \;=\; z^\mu_A(\tau,\vec{\sigma})\,\tilde{A}_\mu(z(\tau,\vec{\sigma}))$ (these potentials know the simultaneity surfaces Σ_τ), whose associated field strength is $F_{AB}(\tau,\vec{\sigma}) = \partial_A\,A_B(\tau,\vec{\sigma}) - \partial_B\,A_A(\tau,\vec{\sigma}) = z^\mu_A(\tau,\vec{\sigma})\,z^\nu_B(\tau,\vec{\sigma})\,\tilde{F}_{\mu\nu}(z(\tau,\vec{\sigma}))$.

The Lagrangian density ($g = |det\, g_{AB}|$, $g_{AB} = z^\mu_A\,\eta_{\mu\nu}\,z^\nu_B$)

$$
\begin{aligned}
\mathcal{L}(\tau,\vec{\sigma}) \;&=\; -\frac{1}{4}\,\sqrt{g(\tau,\vec{\sigma})}\,g^{AC}(\tau,\vec{\sigma})\,g^{BD}(\tau,\vec{\sigma})\,F_{AB}(\tau,\vec{\sigma})\,F_{CD}(\tau,\vec{\sigma}),\\
&\Rightarrow S = \int d\tau d^3\sigma\,\mathcal{L}(\tau,\vec{\sigma}) = \int d\tau\,L(\tau),
\end{aligned}
\tag{6.71}
$$

leads to the canonical momenta [47]

$$
\begin{aligned}
\pi^\tau(\tau,\vec{\sigma}) \;&=\; \frac{\partial\mathcal{L}(\tau,\vec{\sigma})}{\partial\partial_\tau A_\tau(\tau,\vec{\sigma})} = 0,\\
\pi^r(\tau,\vec{\sigma}) \;&=\; \frac{\partial\mathcal{L}(\tau,\vec{\sigma})}{\partial\partial_\tau A_r(\tau,\vec{\sigma})} = -\frac{\gamma(\tau,\vec{\sigma})}{\sqrt{g(\tau,\vec{\sigma})}}\gamma^{rs}(\tau,\vec{\sigma})\,(F_{\tau\ s} - g_{\tau v}\,\gamma^{vu}\,F_{us})(\tau,\vec{\sigma}) =\\
&=\; \frac{\gamma(\tau,\vec{\sigma})}{\sqrt{g(\tau,\vec{\sigma})}}\,\gamma^{rs}(\tau,\vec{\sigma})\,(E_s(\tau,\vec{\sigma}) + g_{\tau v}(\tau,\vec{\sigma})\,\gamma^{vu}(\tau,\vec{\sigma})\,\epsilon_{ust}\,B_t(\tau,\vec{\sigma})),\\
\rho_\mu(\tau,\vec{\sigma}) \;&=\; -\frac{\partial\mathcal{L}(\tau,\vec{\sigma})}{\partial z^\mu_\tau(\tau,\vec{\sigma})} =\\
&=\; \frac{\sqrt{g(\tau,\vec{\sigma})}}{4}\,[(g^{\tau\tau}\,z_{\tau\mu} + g^{\tau r}\,z_{r\mu})(\tau,\vec{\sigma})\,g^{AC}(\tau,\vec{\sigma})\,g^{BD}(\tau,\vec{\sigma})\,F_{AB}(\tau,\vec{\sigma})\,F_{CD}(\tau,\vec{\sigma}) -\\
&-\; 2\,[z_{\tau\mu}(\tau,\vec{\sigma})\,(g^{A\tau}\,g^{\tau C}\,g^{BD} + g^{AC}\,g^{B\tau}\,g^{\tau D})(\tau,\vec{\sigma}) +\\
&+\; z_{\check{r}\mu}(\tau,\vec{\sigma})\,(g^{Ar}\,g^{\tau C} + g^{A\tau}\,g^{rC})(\tau,\vec{\sigma})\,g^{BD}(\tau,\vec{\sigma})]\,F_{AB}(\tau,\vec{\sigma})\,F_{CD}(\tau,\vec{\sigma})] =
\end{aligned}
$$

[47] $\gamma = |det\, g_{rs}|$, $E_r = F_{r\tau}$ $(= -\boldsymbol{\epsilon}\,E^r\; on\, hyper-planes)$, $B_r = \frac{1}{2}\,\epsilon_{ruv}\,F_{uv}$.

$$= [(\rho_\nu\, l^\nu)\, l_\mu + (\rho_\nu\, z^\nu_r)\, \gamma^{rs}\, z_{s\mu}](\tau,\vec\sigma),$$

$$\begin{aligned}
&\{z^\mu(\tau,\vec\sigma), \rho_\nu(\tau,\vec\sigma_1)\} = -\delta^\mu_\nu\, \delta^3(\vec\sigma-\vec\sigma_1),\\
&\{A_A(\tau,\vec\sigma), \pi^B(\tau,\vec\sigma_1)\} = \delta^B_A\, \delta^3(\vec\sigma-\vec\sigma_1),
\end{aligned} \tag{6.72}$$

to the canonical Hamiltonian $H_c = -\int d^3\sigma\, A_\tau(\tau,\vec\sigma)\,\Gamma(\tau,\vec\sigma)$ with $\Gamma(\tau,\vec\sigma) = \partial_r\,\pi^r(\tau,\vec\sigma)$ and to the following generators of the Poincare' group [the suffix "s" is for surface]

$$\begin{aligned}
P^\mu_s &= \int d^3\sigma\, \rho^\mu(\tau,\vec\sigma),\\
J^{\mu\nu}_s &= \int d^3\sigma\, (z^\mu(\tau,\vec\sigma)\rho^\nu(\tau,\vec\sigma) - z^\nu(\tau,\vec\sigma)\rho^\mu(\tau,\vec\sigma)).
\end{aligned} \tag{6.73}$$

There are five primary and one secondary first class constraints

$$\begin{aligned}
\pi^\tau(\tau,\vec\sigma) &\approx 0,\\
\Gamma(\tau,\vec\sigma) &= \partial_r\,\pi^r(\tau,\vec\sigma) \approx 0,
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_\mu(\tau,\vec\sigma) &= \rho_\mu(\tau,\vec\sigma) - l_\mu(\tau,\vec\sigma)\, T_{\tau\tau}(\tau,\vec\sigma) - z_{r\mu}(\tau,\vec\sigma)\, \gamma^{rs}(\tau,\vec\sigma)\, T_{\tau s}(\tau,\vec\sigma) \stackrel{def}{=}\\
&\stackrel{def}{=} \rho_\mu(\tau,\vec\sigma) - \mathcal{G}_\mu[z^\mu_r(\tau,\vec\sigma);\, A_r(\tau,\vec\sigma), \pi^s(\tau,\vec\sigma)] \approx 0,
\end{aligned}$$

$$T_{\tau\tau}(\tau,\vec\sigma) = -\frac{1}{2}\,(\frac{1}{\sqrt{\gamma}}\pi^r g_{rs}\pi^s - \frac{\sqrt{\gamma}}{2}\,\gamma^{rs}\,\gamma^{uv}\, F_{ru}\, F_{sv})(\tau,\vec\sigma),$$

$$T_{\tau s}(\tau,\vec\sigma) = F_{st}(\tau,\vec\sigma)\,\pi^t(\tau,\vec\sigma) = \epsilon_{stu}\,\pi^t(\tau,\vec\sigma)\, B_u(\tau,\vec\sigma) = -[\vec\pi(\tau,\vec\sigma)\times\vec B(\tau,\vec\sigma)]_s,$$

$$\begin{aligned}
\{\mathcal{H}_\mu(\tau,\vec\sigma) \quad &, \quad \mathcal{H}_\nu(\tau,\vec\sigma^{'})\} = \{[l_\mu(\tau,\vec\sigma)\, z_{r\nu}(\tau,\vec\sigma) - l_\nu(\tau,\vec\sigma)\, z_{r\mu}(\tau,\vec\sigma)]\, \frac{\pi^r(\tau,\vec\sigma)}{\sqrt{\gamma(\tau,\vec\sigma)}} -\\
&- \quad z_{u\mu}(\tau,\vec\sigma)\,\gamma^{ur}(\tau,\vec\sigma)\, F_{rs}(\tau,\vec\sigma)\,\gamma^{sv}(\tau,\vec\sigma)\, z_{v\nu}(\tau,\vec\sigma)\}\,\Gamma(\tau,\vec\sigma)\,\delta^3(\vec\sigma-\vec\sigma^{'}) \approx 0.
\end{aligned} \tag{6.74}$$

The constraints $\pi^\tau(\tau,\vec\sigma)\approx 0$ and $\Gamma(\tau,\vec\sigma)\approx 0$ are the canonical generators of the electromagnetic gauge transformations, while $\mathcal{H}_\mu(\tau,\vec\sigma)\approx 0$ generate the gauge transformations

from an admissible 3+1 splitting of Minkowski space-time to another one with the associated change in the notion of simultaneity.

Instead of the constraints $\mathcal{H}_\mu(\tau,\vec{\sigma}) \approx 0$, we can use their projections $\mathcal{H}_r(\tau,\vec{\sigma}) = \mathcal{H}^\mu(\tau,\vec{\sigma})\, z_{r\mu}(\tau,\vec{\sigma}) \approx 0$, $\mathcal{H}_\perp(\tau,\vec{\sigma}) = \mathcal{H}^\mu(\tau,\vec{\sigma})\, l_\mu(\tau,\vec{\sigma}) \approx 0$, normal and tangent to the simultaneity surfaces Σ_τ. Modulo the Gauss law constraint $\Gamma(\tau,\vec{\sigma}) \approx 0$, the new constraints satisfy the universal Dirac algebra of the superhamiltonian and supermomentum constraints of canonical metric gravity (see Ref.[88]). This implies [88] that the gauge transformations generated by the constraint $\mathcal{H}_\perp(\tau,\vec{\sigma})$ change the form of the simultaneity surfaces Σ_τ (i.e. the 3+1 splitting), while those generated by the constraints $\mathcal{H}_r(\tau,\vec{\sigma})$ change the 3-coordinates on such surfaces.

The Dirac Hamiltonian and the Hamilton-Dirac equations are ($\lambda^\mu(\tau,\vec{\sigma})$ and $\lambda_\tau(\tau,\vec{\sigma})$ are arbitrary Dirac multipliers; $\stackrel{\circ}{=}$ means evaluated on the extremals of the action principle)

$$
\begin{aligned}
H_D &= \int d^3\sigma\, [\lambda^\mu(\tau,\vec{\sigma})\, \mathcal{H}_\mu(\tau,\vec{\sigma}) + \lambda_\tau(\tau,\vec{\sigma})\, \pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\, \Gamma(\tau,\vec{\sigma})] = \\
&= \int d^3\sigma\, [\tilde{\lambda}_\perp(\tau,\vec{\sigma})\, \mathcal{H}_\perp(\tau,\vec{\sigma}) + \\
&+ \tilde{\lambda}^r(\tau,\vec{\sigma})\, \mathcal{H}_r(\tau,\vec{\sigma}) + \lambda_\tau(\tau,\vec{\sigma})\, \pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\, \Gamma(\tau,\vec{\sigma})],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A_\tau(\tau,\vec{\sigma})}{\partial\tau} &\stackrel{\circ}{=} \{A_\tau(\tau,\vec{\sigma}), H_D\} = \lambda_\tau(\tau,\vec{\sigma}), \\
\frac{\partial A_r(\tau,\vec{\sigma})}{\partial\tau} &\stackrel{\circ}{=} \{A_r(\tau,\vec{\sigma}), H_D\} = -\int d^3\sigma^{'}\, \Big[\lambda^\mu(\tau,\vec{\sigma}^{'})\, \{A_r(\tau,\vec{\sigma}), \\
&\quad \mathcal{G}_\mu(z^\alpha(\tau,\vec{\sigma}^{'}), A_u(\tau,\vec{\sigma}^{'}), \pi^v(\tau,\vec{\sigma}^{'}))\} + \\
&+ A_\tau(\tau,\vec{\sigma}^{'})\, \{A_r(\tau,\vec{\sigma}), \Gamma(\tau,\vec{\sigma}^{'})\}\Big], \\
\frac{\partial \pi^r(\tau,\vec{\sigma})}{\partial\tau} &\stackrel{\circ}{=} \{\pi^r(\tau,\vec{\sigma}), H_D\} = \\
&= -\int d^3\sigma^{'}\, \lambda^\mu(\tau,\vec{\sigma}^{'})\, \{\pi^r(\tau,\vec{\sigma}), \mathcal{G}_\mu(z^\alpha(\tau,\vec{\sigma}^{'}), A_u(\tau,\vec{\sigma}^{'}), \pi^v(\tau,\vec{\sigma}^{'}))\},
\end{aligned}
\tag{6.75}
$$

Due to the last two lines of Eqs.(6.74), we see that two successive gauge transformations, of generators $G_i(\tau,\vec{\sigma}) = \lambda_i^\mu(\tau,\vec{\sigma})\, \mathcal{H}_\mu(\tau,\vec{\sigma})$, $i = 1,2$, do not commute but imply an electro-

magnetic gauge transformation. Since the effect of the $i = 1, 2$ gauge transformations is to modify the notions of simultaneity, also the definition of the Dirac observables of the electro-magnetic field will change with the 3+1 splitting. In general, given two different 3+1 splittings, the two sets of Dirac observables associated with them will be connected by an electro-magnetic gauge transformation.

Since it is not clear whether it is possible to find a quasi-Shanmugadhasan canonical transformation adapted to $\mathcal{H}_r(\tau, \vec{\sigma}) = \mathcal{H}_\mu(\tau, \vec{\sigma})\, z^\mu_r(\tau, \vec{\sigma}) \approx 0$, $\pi^\tau(\tau, \vec{\sigma}) \approx 0$, $\Gamma(\tau, \vec{\sigma}) \approx 0$ [48], the search of the electro-magnetic Dirac observables must be done with the following strategy:

i) make the choice of an admissible 3+1 splitting by adding four gauge-fixing constraints determining the embedding $z^\mu(\tau, \vec{\sigma})$, so that the induced 4-metric $g_{AB}(\tau, \vec{\sigma})$ becomes a numerical quantity and is no more a configuration variable;

ii) find the Dirac observables on the resulting completely fixed simultaneity surfaces Σ_τ with a suitable Shanmugadhasan canonical transformation adapted to the two remaining electro-magnetic constraints.

Let us remark that a similar scheme has to be followed also in the canonical Einstein-Maxwell theory: only after having fixed a 3+1 splitting (a system of 4-coordinates on the solutions of Einstein's equations) we can find the Dirac observables of the electro-magnetic field.

This strategy is induced by the fact that, while the Gauss law constraint $\Gamma(\tau, \vec{\sigma}) = \partial_r\, \pi^r(\tau, \vec{\sigma}) \approx 0$ is a scalar under change of admissible 3+1 splittings [49], the gauge vector potential $A_r(\tau, \vec{\sigma})$ is the pull-back to the base of a connection one-form and can be considered as a tensor only with topologically trivial surfaces Σ_τ (like in the case we are considering). Since a Shanmugadhasan canonical transformation adapted to the Gauss law constraint transforms $\Gamma(\tau, \vec{\sigma})$ in one of the new momenta, it is not clear how to define a conjugate gauge variable $\eta_{em}(\tau, \vec{\sigma})$ such that $\{\eta_{em}(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}_1)\} = \delta^3(\vec{\sigma}, \vec{\sigma}_1)$ and two conjugate pairs of Dirac observables having vanishing Poisson brackets with both $\eta_{em}(\tau, \vec{\sigma})$ and $\Gamma(\tau, \vec{\sigma})$ when the 3-metric on Σ_τ is not Euclidean ($g_{rs}(\tau, \vec{\sigma}) \neq -\boldsymbol{\epsilon}\, \delta_{rs}$).

The only case studied till now [89] is the restriction to the Wigner hyper-planes associated to the rest-frame instant form, where both $A_r(\tau, \vec{\sigma})$ and $\pi^r(\tau, \vec{\sigma})$ transform as Wigner spin-1

[48] $\mathcal{H}_\perp(\tau, \vec{\sigma}) = \mathcal{H}^\mu(\tau, \vec{\sigma})\, l_\mu(\tau, \vec{\sigma}) \approx 0$, like an ordinary Hamiltonian, can be included in the adapted Darboux-Shanmugadhasan basis only in case of integrability of the equations of motion.

[49] $\pi^r(\tau, \vec{\sigma})$ is a vector density like in canonical metric gravity.

3-vectors under Lorentz boosts. Since on Wigner hyper-planes we have $g_{rs}(\tau, \vec{\sigma}) = -\boldsymbol{\epsilon}\, \delta_{rs}$, the Shanmugadhasan canonical transformation leads to a *radiation gauge*

$$
\begin{array}{|c|}\hline A_A \\ \hline \pi^A \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|}\hline A_\tau & \eta_{em} & A_{\perp\, r} \\ \hline \pi^\tau \approx 0 & \Gamma \stackrel{\circ}{\approx} 0 & \pi^r_\perp \\ \hline \end{array}
$$

$$
\begin{aligned}
A^r(\tau, \vec{\sigma}) &= -\boldsymbol{\epsilon}\, A_r(\tau, \vec{\sigma}) = \frac{\partial}{\partial \sigma^r} \eta_{em}(\tau, \vec{\sigma}) + A^r_\perp(\tau, \vec{\sigma}), \\
\pi^r(\tau, \vec{\sigma}) &= \pi^r_\perp(\tau, \vec{\sigma}) + \frac{1}{\triangle_\sigma} \frac{\partial}{\partial \sigma^r}\, \Gamma(\tau, \vec{\sigma}), \qquad \triangle_\sigma = -\vec{\partial}^2_\sigma, \\
\eta_{em}(\tau, \vec{\sigma}) &= -\frac{1}{\triangle_\sigma} \frac{\partial}{\partial \vec{\sigma}} \cdot \vec{A}(\tau, \vec{\sigma}), \\
A^r_\perp(\tau, \vec{\sigma}) &= (\delta^{rs} + \frac{\partial^r_\sigma \partial^s_\sigma}{\triangle_\sigma})\, A_s(\tau, \vec{\sigma}), \\
\pi^r_\perp(\tau, \vec{\sigma}) &= (\delta^{rs} + \frac{\partial^r_\sigma \partial^s_\sigma}{\triangle_\sigma})\, \pi_s(\tau, \vec{\sigma}), \qquad \vec{\pi}^2(\tau, \vec{\sigma}) \approx \vec{\pi}^2_\perp(\tau, \vec{\sigma}),
\end{aligned}
$$

$$
\{\eta_{em}(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}^{'})\}^{**} = -\boldsymbol{\epsilon}\, \delta^3(\vec{\sigma} - \vec{\sigma}^{'}),
$$

$$
\{A^r_\perp(\tau, \vec{\sigma}), \pi^s_\perp(\tau, \vec{\sigma}^{'})\}^{**} = -\boldsymbol{\epsilon}\, (\delta^{rs} + \frac{\partial^r_\sigma \partial^s_\sigma}{\triangle_\sigma}) \delta^3(\vec{\sigma} - \vec{\sigma}^{'}). \qquad (6.76)
$$

With every fixed type of simultaneity surface Σ_τ with non-trivial 3-metric, $g_{rs}(\tau, \vec{\sigma}) \neq -\boldsymbol{\epsilon}\, \delta_{rs}$, we have to find suitable gauge variable $\eta_{em}(\tau, \vec{\sigma})$ and the Dirac observables replacing $A^r_\perp(\tau, \vec{\sigma})$ and $\pi^r_\perp(\tau, \vec{\sigma})$.

To choose a 3+1 splitting with a foliation, whose simultaneity surfaces are described by a given admissible embedding $z^\mu_F(\tau, \vec{\sigma})$, we add the gauge-fixing constraints

$$
\zeta^\mu(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma}) - z^\mu_F(\tau, \vec{\sigma}) \approx 0,
$$

$$
z^\mu_F(\tau, \vec{\sigma}) = x^\mu(\tau) + F^\mu(\tau, \vec{\sigma}) \;\rightarrow_{|\vec{\sigma}| \rightarrow \infty}\; x^\mu_{(\infty)}(0) + l^\mu_F\, \tau + \epsilon^\mu_{F\, r}\, \sigma^r \stackrel{def}{=} x^\mu_{(\infty)}(\tau) + \epsilon^\mu_{F\, r}\, \sigma^r,
$$

$$x^\mu(\tau) = l^\mu_F\, a(\tau) - \epsilon^\mu_{F\, r}\, b^r(\tau), \qquad F^\mu(\tau, \vec{0}) = 0,$$

$$F^\mu(\tau, \vec{\sigma}) \,\rightarrow_{|\vec{\sigma}| \rightarrow \infty}\, l^\mu_F\, [\tau - a(\tau)] + \epsilon^\mu_{F\, r}\, [\sigma^r + b^r(\tau)],$$

$$z^\mu_{F\, \tau}(\tau, \vec{\sigma}) = \dot{x}^\mu(\tau) + \frac{\partial F^\mu(\tau, \vec{\sigma})}{\partial \tau} \,\rightarrow_{|\vec{\sigma}| \rightarrow \infty}\, \dot{x}^\mu_{(\infty)}(\tau) = l^\mu_F,$$

$$det\, \Big(\{ \zeta^\mu(\tau, \vec{\sigma}), \mathcal{H}_\nu(\tau, \vec{\sigma}_1) \} \Big) \not= 0, \tag{6.77}$$

where we have used the notation of Eqs.(3.3), (3.5). Here $\epsilon^\mu_{F\, A} = \Big(l^\mu_F = \epsilon^\mu_{F\, \tau}; \epsilon^\mu_{F\, r} \Big)$ is the asymptotic tetrad at spatial infinity associated to the foliation, with l^μ_F the normal to the asymptotic hyper-planes (see Section I). If $l^\mu_{(F)}(\tau, \vec{\sigma})$ are the components of the unit normal vector field to $\Sigma_{F\, \tau}$, built in terms of the $z^\mu_{F\, r}(\tau, \vec{\sigma})$, we have $l^\mu_{(F)}(\tau, \vec{\sigma}) \,\rightarrow_{|\vec{\sigma}| \rightarrow \infty}\, l^\mu_F$.

The functions $F^\mu(\tau, \vec{\sigma})$, assumed to satisfy the admissibility conditions of Section III, describe the form of the simultaneity surfaces $\Sigma_{F\, \tau}$, while the arbitrary centroid $x^\mu(\tau)$ (a time-like, in general non-inertial, observer chosen as origin of the 3-coordinates on $\Sigma_{F\, \tau}$) describes how they are packed in the foliation. The centroid corresponds to an observer of the rotating skew congruence associated to the foliation, because $\dot{x}^\mu(\tau) = z^\mu_{F\, \tau}(\tau, \vec{0})$. Instead $x^\mu_{(\infty)}(\tau)$ is the world-line of an asymptotic inertial observer at spatial infinity.

For instance, if we want to select foliations which arise from those admitted in the rest-frame instant form of canonical metric gravity [88] in the limit of vanishing gravitational field, we must have that the hyper-surfaces Σ_τ tend in a direction-independent way to Wigner hyper-planes at spatial infinity, $z^\mu_W(\tau, \vec{\sigma}) = x^\mu(0) + \epsilon^\mu_A(u(P_s))\, \sigma^A$. This can be obtained with admissible embeddings of the type $z^\mu_F(\tau, \vec{\sigma}) = x^\mu(0) + \epsilon^\mu_A(u(P))\, [\sigma^A + F^A(\tau, \vec{\sigma})]$, $lim_{|\vec{\sigma}| \rightarrow \infty}\, F^A(\tau, \vec{\sigma}) = 0$. Here P^μ is a time-like 4-vector and $\epsilon^\mu_A(u(P)) = \Big(u^\mu(P) = P^\mu / \sqrt{\boldsymbol{\epsilon}\, P^2}; \epsilon^\mu_r(u(P)) \Big)$ is the tetrad defined by the standard Wigner boost sending P^μ at rest. To enforce the requirement $P^\mu = P^\mu_s$ (asymptotism to Wigner hyper-planes), we enlarge the phase space with four pairs X^μ, P^μ of conjugate variables and we add the four first class constraints $P^\mu - P^\mu_s \approx 0$ to the Dirac Hamiltonian. After having evaluated the Dirac brackets associated to the gauge fixings (6.77), the 4-metric $g_{AB}(\tau, \vec{\sigma})$ becomes a function $g_{(F) AB}(\tau, \vec{\sigma} | P)$ depending only on P^μ.

In special relativity a simpler set of admissible embeddings is given by the foliations with

parallel space-like hyper-planes of Section IV. Before restricting to them let us delineate how the method of canonical reduction to Wigner hyper-planes introduced in Ref.[89] is modified when there is a gauge fixing (6.77) restricting the description to specific admissible simultaneity surfaces.

The preservation in time of the gauge fixing (6.77) gives the following determination of the Dirac multipliers $\lambda^\mu(\tau,\vec{\sigma})$ appearing in the Dirac Hamiltonian (6.75)

$$\frac{\partial \zeta^\mu(\tau,\vec{\sigma})}{\partial\tau} \,\stackrel{\circ}{=}\, \{z^\mu(\tau,\vec{\sigma}), H_D\} - \frac{\partial z^\mu_F(\tau,\vec{\sigma})}{\partial\tau} \approx 0,$$

$$\Downarrow$$

$$\begin{aligned}
\lambda^\mu(\tau,\vec{\sigma}) &= \lambda^\mu(\tau) - \frac{\partial F^\mu(\tau,\vec{\sigma})}{\partial\tau}, \qquad \lambda^\mu(\tau) = -\dot{x}^\mu(\tau) = -z^\mu_{F\,\tau}(\tau,\vec{0}),\\
H_D &= \lambda^\mu(\tau)\, H_\mu(\tau) + \int d^3\sigma\, \Big[-\frac{\partial F^\mu}{\partial\tau}\, \mathcal{H}_\mu + \lambda_\tau\, \pi^\tau - A_\tau\, \Gamma\Big](\tau,\vec{\sigma}),\\
&\quad H_\mu(\tau) = \int d^3\sigma\, \mathcal{H}_\mu(\tau,\vec{\sigma}),
\end{aligned} \tag{6.78}$$

with the arbitrariness reduced to $\lambda^\mu(\tau)$.

If we go to the reduced phase space by introducing the Dirac brackets associated to the gauge fixing (6.77), we see that all the gauge degrees of freedom of the embedding are reduced to the four variables $x^\mu(\tau)$. Therefore the first class constraints $\mathcal{H}_\mu(\tau,\vec{\sigma})$ are reduced to only four independent ones

$$\begin{aligned}
H^\mu(\tau) &= P^\mu_s - \int d^3\sigma\, [l_{(F)\mu}(\tau,\vec{\sigma})\, T_{(F)\tau\tau}(\tau,\vec{\sigma}) + z_{Fr\mu}(\tau,\vec{\sigma})\, \gamma^{rs}_F(\tau,\vec{\sigma})\, T_{(F)\tau s}(\tau,\vec{\sigma})] =\\
&= P^\mu_s - \int d^3\sigma\, \mathcal{G}_\mu\, [z^\mu_{F\,r}(\tau,\vec{\sigma});\; A_r(\tau,\vec{\sigma}), \pi^s(\tau,\vec{\sigma})] \approx 0,
\end{aligned} \tag{6.79}$$

where $P^\mu_s(\tau)$ is the canonical momentum conjugate to $x^\mu(\tau)$

$$\{x^\mu(\tau), P^\nu_s\}^* = -\eta^{\mu\nu}. \tag{6.80}$$

Let us remark that the Dirac brackets for the electro-magnetic field and their momenta remain equal to their Poisson brackets.

If we restrict Hamilton equations (6.75) to the gauge (6.77) and we use Eqs.(6.78), we get a form of these equations, which can be reproduced as the Hamilton equations of the reduced phase space only by using the new Dirac Hamiltonian (it differs from H_D of Eqs.(6.78) by the term $-\int d^3\sigma\, \rho_\mu(\tau,\vec{\sigma})\, \frac{\partial F^\mu(\tau,\vec{\sigma})}{\partial \tau}$, which is ineffective in the reduced phase space)

$$\begin{aligned}\tilde{H}_D \,&=\, \lambda^\mu(\tau)\, H_\mu(\tau) + \int d^3\sigma\, \frac{\partial F^\mu(\tau,\vec{\sigma})}{\partial \tau}\, \mathcal{G}_\mu\, [z^\mu_{F\,r}(\tau,\vec{\sigma});\, A_r(\tau,\vec{\sigma}), \pi^s(\tau,\vec{\sigma})] + \\ &+\, \int d^3\sigma\, [\lambda_\tau(\tau,\vec{\sigma})\, \pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\, \Gamma(\tau,\vec{\sigma})]. \qquad (6.81)\end{aligned}$$

The induced 4-metric $g_{(F)AB}(\tau,\vec{\sigma}) \,=\, z^\mu_{F\,A}(\tau,\vec{\sigma})\, \eta_{\mu\nu}\, z^\nu_{F\,B}(\tau,\vec{\sigma})$ is completely determined except for its dependence on the arbitrary velocity of the centroid, $\dot{x}^\mu(\tau) = -\lambda^\mu(\tau)$. The constraints (6.79) determine the generator of translations P^μ_s, given in Eq.(6.73), so that the *coordinates of the centroid are gauge variables*, corresponding to the *arbitrariness in the choice of the (non-inertial) observer*.

From the second line of Eq.(6.73) we get the following form of the generator of Lorentz transformations

$$\begin{aligned}J^{\mu\nu}_s \,&=\, x^\mu\, P^\nu_s - x^\nu\, P^\mu_s + S^{\mu\nu}_s,\\ S^{\mu\nu}_s \,&=\, \int d^3\sigma\, \Big[F^\mu\, \mathcal{G}^\nu\, [z^\mu_{F\,r};\, A_r, \pi^s] - F^\nu\, \mathcal{G}^\nu\, [z^\mu_{F\,r};\, A_r, \pi^s]\Big](\tau,\vec{\sigma}). \qquad (6.82)\end{aligned}$$

By using the asymptotic tetrad at spatial infinity the four constraints $H^\mu(\tau) \approx 0$ and the Dirac Hamiltonian may be transformed in the following form

$$\begin{aligned}\bar{H}_l(\tau) \,&=\, l_{F\,\mu}\, P^\mu_s - l_{F\,\mu} \int d^3\sigma\, [l^\mu_{(F)}\, T_{(F)\tau\tau} - z^\mu_{F\,r}\, \gamma^{rs}_F\, T_{(F)\tau s}](\tau,\vec{\sigma}) = \\ &=\, l_{F\,\mu}\, P^\mu_s - l_{F\,\mu} \int d^3\sigma\, \mathcal{G}^\mu\, [z^\mu_{F\,r}(\tau,\vec{\sigma});\, A_r(\tau,\vec{\sigma}), \pi^s(\tau,\vec{\sigma})] = l_F \cdot P_s - M_{(F)l} \approx 0,\\ \bar{H}_r(\tau) \,&=\, \epsilon_{F\,r\mu}\, P^\mu_s - \epsilon_{F\,r\mu} \int d^3\sigma\, \mathcal{G}^\mu\, [z^\mu_{F\,r}(\tau,\vec{\sigma});\, A_r(\tau,\vec{\sigma}), \pi^s(\tau,\vec{\sigma})](\tau,\vec{\sigma}) \approx 0,\end{aligned}$$

$$
\begin{aligned}
\bar{H}_D &= \bar{\lambda}_l(\tau)\, \bar{H}_l(\tau) - \sum_r \bar{\lambda}_r(\tau)\, \bar{H}_r(\tau) + \\
&+ \int d^3\sigma\, \frac{\partial F^\mu(\tau,\vec{\sigma})}{\partial \tau}\, \mathcal{G}_\mu\, [z^\mu_{F\,r}(\tau,\vec{\sigma});\, A_r(\tau,\vec{\sigma}), \pi^s(\tau,\vec{\sigma})] + \\
&+ \int d^3\sigma\, [\lambda_\tau(\tau,\vec{\sigma})\, \pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\, \Gamma(\tau,\vec{\sigma})]. \qquad (6.83)
\end{aligned}
$$

Since $l_F \cdot x(\tau)$, $l_F \cdot P_s$ and $\epsilon_{F\,r} \cdot x(\tau)$, $\epsilon_{F\,r} \cdot P_s$ are four pairs of conjugate variables, we see that

i) the gauge fixing $l_F \cdot x(\tau) - \tau \approx 0$ (so that $a(\tau) = \boldsymbol{\epsilon}\, \tau$) to $\bar{H}_l(\tau) \approx 0$ identifies the mathematical time τ with the Lorentz-scalar asymptotic time of the asymptotic inertial observer $x^\mu_{(\infty)}(\tau)$and, through Dirac brackets, forces the identity $l_F \cdot P_s \equiv M_{(F)l}$, with $M_{(F)l} = l_{F\,\mu} \int d^3\sigma\, \mathcal{G}^\mu\, [z^\mu_{F\,r}(\tau,\vec{\sigma});\, A_r(\tau,\vec{\sigma}), \pi^s(\tau,\vec{\sigma})]$ being the *internal mass* seen by the observer $x^\mu(\tau)$;

ii) the gauge fixings $\epsilon_{F\,r} \cdot x(\tau) - g_r(\tau) \approx 0$ (so that $b^r(\tau) = \boldsymbol{\epsilon}\, g_r(\tau)$ and $\lambda^\mu(\tau) = -\dot{x}^\mu(\tau) = -\boldsymbol{\epsilon}\, [l^\mu_F - \epsilon^\mu_{F\,r}\, \dot{g}_r(\tau)]$) to $\bar{H}_r(\tau) \approx 0$ identify the centroid $x^\mu(\tau)$ with the world-line $\tilde{x}^\mu(\tau) = \boldsymbol{\epsilon}\, [l^\mu_F\, \tau - \epsilon^\mu_{F\,r}\, g_r(\tau)]$ of a time-like (in general non-inertial) observer, whose *3-momentum* is $\epsilon_{F\,r} \cdot P_s \equiv \epsilon_{F\,r\,\mu} \int d^3\sigma\, \mathcal{G}^\mu\, [z^\nu_{F\,s}(\tau,\vec{\sigma});\, A_s(\tau,\vec{\sigma}), \pi^s(\tau,\vec{\sigma})]$. It plays the same role of the *external 4-center of mass* of the rest-frame instant form [43]. For $g_r(\tau) = 0$, this observer coincides with asymptotic inertial one: $\tilde{x}^\mu(\tau) = \boldsymbol{\epsilon}\, x^\mu_{(\infty)}(\tau)$.

The *internal angular momentum* is $S_{(F)s\,AB} = \epsilon_{F\,A\,\mu}\, \epsilon_{F\,B\,\nu}\, S^{\mu\nu}_s$. This quantity, $M_{(F)l}$ and $\epsilon_{F\,r} \cdot P_s$, replace the generators of the internal Poincare' group of the rest-frame instant form on Wigner hyper-planes [43, 89].

After the previous gauge fixings we arrive at a phase space containing only the electro-magnetic field restricted to the simultaneity surfaces $\Sigma_{F\,\tau}$ of the completely fixed embedding $z^\mu_F(\tau,\vec{\sigma}) = \tilde{x}^\mu(\tau) + F^\mu(\tau,\vec{\sigma}) \;\rightarrow |\vec{\sigma}| \rightarrow \infty\, x^\mu_{(\infty)}(\tau) + \epsilon^\mu_{F\,r}\, \sigma^r$ and with a non-vanishing Dirac Hamiltonian given by the last two lines of $\bar{H}_D$ in Eq.(6.83). However, since the gauge fixings are explicitly τ-dependent, this restricted Hamiltonian does not reproduce the same Hamilton equations for the electro-magnetic field that would be obtained by using $\bar{H}_D$ of Eq.(6.83) and then restricting them with the gauge fixings. The same steps used to get

Eq.(6.81) show that the true Hamiltonian $\hat{H}_D$ acting in the reduced phase space is obtained by adding the projection $\dot{\tilde{x}}_\mu(\tau)\int d^3\sigma\,\mathcal{G}^\mu(..)(\tau,\vec{\sigma})$ of the total 4-momentum along the 4-velocity $\dot{\tilde{x}}^\mu(\tau) = -\lambda^\mu(\tau)$ of the observer $\tilde{x}^\mu(\tau)$ to the reduced Dirac-Hamiltonian. The final Hamiltonian $\hat{H}_D$ is the sum of an *effective non-inertial Hamiltonian* (containing the internal mass and the internal 3-momentum) and of the generator of the electro-magnetic gauge transformations

$$\begin{aligned}
\hat{H}_D &= \int d^3\sigma\, z^\mu_{F\tau}(\tau,\vec{\sigma})\,\mathcal{G}_\mu\,[z^\mu_{F\,r}(\tau,\vec{\sigma});\; A_r(\tau,\vec{\sigma}),\pi^s(\tau,\vec{\sigma})] +\\
&+ \int d^3\sigma\,[\lambda_\tau(\tau,\vec{\sigma})\,\pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\,\Gamma(\tau,\vec{\sigma})] =\\
&= M_{(F)l} + \int d^3\sigma\,\Big[\epsilon^\mu_{F\,r}\,\dot{g}_r(\tau) + \frac{\partial F^\mu(\tau,\vec{\sigma})}{\partial\tau}\Big]\,\mathcal{G}_\mu\,[z^\mu_{F\,r}(\tau,\vec{\sigma});\; A_r(\tau,\vec{\sigma}),\pi^s(\tau,\vec{\sigma})] +\\
&+ \int d^3\sigma\,[\lambda_\tau(\tau,\vec{\sigma})\,\pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\,\Gamma(\tau,\vec{\sigma})] =\\
&= M_{(F)l} + \dot{g}_r(\tau)\,\epsilon_{F\,r}\cdot P_s + \int d^3\sigma\,\frac{\partial F^\mu(\tau,\vec{\sigma})}{\partial\tau}\,\mathcal{G}_\mu\,[z^\mu_{F\,r}(\tau,\vec{\sigma});\; A_r(\tau,\vec{\sigma}),\pi^s(\tau,\vec{\sigma})] +\\
&+ \int d^3\sigma\,[\lambda_\tau(\tau,\vec{\sigma})\,\pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\,\Gamma(\tau,\vec{\sigma})]. \qquad (6.84)
\end{aligned}$$

This is the generator of the evolution seen by the non-inertial observer $\tilde{x}^\mu(\tau) = \boldsymbol{\epsilon}\,[l^\mu_F\,\tau - \epsilon^\mu_{F\,r}\,g_r(\tau)]$ as a consequence of the chosen notion of simultaneity with its asymptotic constant tetrad ϵ^μ_A at spatial infinity.

Therefore the time-like non-inertial observer (not orthogonal to the instantaneous 3-space $\Sigma_{F\,\tau}$) $x^\mu(\tau) \equiv \tilde{x}^\mu(\tau)$ with $\dot{x}^\mu(\tau) = z^\mu_{F\tau}(\tau,\vec{0})$ must

i) use the 3+1 point of view (instantaneous 3-space $\Sigma_{F\,\tau}$) to describe the evolution in $\tau \equiv l_F\cdot\tilde{x}$ with $\hat{H}_D$ as Hamiltonian: besides an electro-magnetic internal mass term, $M_{(F)l}$, like in the rest-frame instant form, there are two extra terms interpretable as potentials of the *inertial forces* associated to this notion of simultaneity;

ii) make the choice of a tetrad $\mathcal{V}^\mu_A = \Big(\mathcal{V}^\mu_\tau = \dot{\tilde{x}}^\mu/\sqrt{\boldsymbol{\epsilon}\,\dot{\tilde{x}}^2};\mathcal{V}^\mu_r\Big)$ and use the 1+3 point of view to measure the tetradic components $\mathcal{F}_{(F)AB} = \mathcal{V}^\mu_A\,\mathcal{V}^\nu_B\,F_{\mu\nu} = \mathcal{V}^\mu_A\,z^C_{F\,\mu}\,\mathcal{V}^\nu_B\,z^D_{F\,\nu}\,F_{CD}$ of the electro-magnetic field.

The Hamilton equations for the vector potential of the electro-magnetic field on the simultaneity surfaces of the foliation $z^\mu_F(\tau,\vec{\sigma}) = \tilde{x}^\mu(\tau) + F^\mu(\tau,\vec{\sigma})$ are

$$\frac{\partial\, A_r(\tau,\vec{\sigma})}{\partial\tau} \stackrel{\circ}{=} \{A_r(\tau,\vec{\sigma}), \hat{H}_D\} = \left[\frac{\sqrt{g_F}}{\gamma_F}\, g_{F\, rs}\, \pi^s + g_{F\,\tau u}\, \gamma_F^{us}\, F_{sr} + \partial_r A_\tau\right](\tau,\vec{\sigma}),$$

$$\frac{\partial\, \pi^r(\tau,\vec{\sigma})}{\partial\tau} \stackrel{\circ}{=} \{\pi^r(\tau,\vec{\sigma}), \hat{H}_D\} = \frac{\partial}{\partial\sigma^s}\, [\sqrt{g_F}\, \gamma_F^{sv}\, \gamma_F^{ru}\, F_{vu} - (g_{F\,\tau u}\, \gamma_F^{us}\, \pi^r - g_{F\,\tau u}\, \gamma_F^{ur}\, \pi^s)]\, (\tau,\vec{\sigma}). \tag{6.85}$$

We can invert the first to obtain

$$\begin{aligned} \pi^s(\tau,\vec{\sigma}) &= -\left[\frac{\gamma_F}{\sqrt{g_F}}\, \gamma_F^{sr}\, (F_{\tau r} - g_{F\,\tau u}\, \gamma_F^{uv}\, F_{vr})\right](\tau,\vec{\sigma}) = \\ &= -\sqrt{g_F(\tau,\vec{\sigma})}\, g_F^{\tau A}(\tau,\vec{\sigma})\, g_F^{sB}(\tau,\vec{\sigma})\, F_{AB}(\tau,\vec{\sigma}). \end{aligned} \tag{6.86}$$

It can be shown that Eqs.(6.85) are equivalent to

$$\frac{\partial}{\partial\sigma^A}\left[\sqrt{g_F(\tau,\vec{\sigma})}\, g_F^{AB}(\tau,\vec{\sigma})\, g_F^{sD}(\tau,\vec{\sigma})\, F_{BD}(\tau,\vec{\sigma})\right] \stackrel{\circ}{=} 0, \tag{6.87}$$

and that by using Eq.(6.86) the Gauss law $\partial_r\, \pi^r(\tau,\vec{\sigma}) = 0$ becomes

$$\frac{\partial}{\partial\sigma^A}\left[\sqrt{g_F(\tau,\vec{\sigma})}\, g_F^{AB}(\tau,\vec{\sigma})\, g_F^{\tau D}(\tau,\vec{\sigma})\, F_{BD}(\tau,\vec{\sigma})\right] \stackrel{\circ}{=} 0. \tag{6.88}$$

As a consequence Eqs.(6.87) and (6.88) imply

$$\frac{1}{\sqrt{g_F(\tau,\vec{\sigma})}}\, \frac{\partial}{\partial\sigma^A}\left[\sqrt{g_F(\tau,\vec{\sigma})}\, g_F^{AB}(\tau,\vec{\sigma})\, g_F^{CD}(\tau,\vec{\sigma})\, F_{BD}(\tau,\vec{\sigma})\right] \stackrel{\circ}{=} 0. \tag{6.89}$$

These are the expected equations for the field strengths written in a *manifestly covariant form*.

If we add to the Lagrangian (6.71) a set of N charged point particles interacting with the electro-magnetic fields (see Refs.[36]) and with 3-positions $\eta_i^r(\tau)$ on $\Sigma_{F\,\tau}$, such that $x_i^\mu(\tau) = z_F^\mu(\tau,\vec{\eta}_i(\tau)$, it can be shown [36] that the Gauss law and the second half of the Hamilton equations (6.85) are modified to the form

$$
\begin{aligned}
\Gamma(\tau,\vec{\sigma}) &= \frac{\partial}{\partial \sigma^r}\,\pi^r(\tau,\vec{\sigma}) - \sum_{i=1}^N Q_i\,\delta(\vec{\sigma}-\vec{\eta}_i(\tau)) \approx 0,\\
\frac{\partial\,\pi^r(\tau,\vec{\sigma})}{\partial \tau} &\stackrel{\circ}{=} \frac{\partial}{\partial \sigma^s}\,[\sqrt{g_F}\,\gamma_F^{sv}\,\gamma_F^{ru}\,F_{vu} - (g_{F\,\tau u}\,\gamma_F^{us}\,\pi^r - g_{F\,\tau u}\,\gamma_F^{ur}\,\pi^s)]\,(\tau,\vec{\sigma}) -\\
&- \sum_{i=1}^N Q_i\,\dot{\eta}^r_i(\tau)\,\delta(\vec{\sigma}-\vec{\eta}_i(\tau)).
\end{aligned}
\tag{6.90}
$$

If we introduce the charge density and the charge current density on Σ_τ

$$
\begin{aligned}
\overline{\rho}(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{\gamma_F(\tau,\vec{\sigma})}}\sum_{i=1}^N Q_i\,\delta(\vec{\sigma}-\vec{\eta}_i(\tau)),\\
\overline{J}^r(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{\gamma_F(\tau,\vec{\sigma})}}\sum_{i=1}^N Q_i\,\dot{\eta}^r_i(\tau)\,\delta(\vec{\sigma}-\vec{\eta}_i(\tau)),
\end{aligned}
\tag{6.91}
$$

so that the total charge is

$$
Q_{tot} = \sum_{i=1}^N = \int d^3\sigma\,\sqrt{\gamma_F(\tau,\vec{\sigma})}\,\overline{\rho}(\tau,\vec{\sigma}),
\tag{6.92}
$$

then Eqs.(6.90) can be rewritten in the more general form

$$
\begin{aligned}
\frac{\partial}{\partial \sigma^r}\,\pi^r(\tau,\vec{\sigma}) &\approx \sqrt{\gamma_F(\tau,\vec{\sigma})}\,\overline{\rho}(\tau,\vec{\sigma}),\\
\frac{\partial\,\pi^r(\tau,\vec{\sigma})}{\partial \tau} &\stackrel{\circ}{=} \frac{\partial}{\partial \sigma^s}\,[\sqrt{g_F}\,\gamma_F^{sv}\,\gamma_F^{ru}\,F_{vu} - (g_{F\,\tau u}\,\gamma_F^{us}\,\pi^r - g_{F\,\tau u}\,\gamma_F^{ur}\,\pi^s)]\,(\tau,\vec{\sigma}) -\\
&- \sqrt{\gamma_F(\tau,\vec{\sigma})}\,\overline{J}^r(\tau,\vec{\sigma}).
\end{aligned}
\tag{6.93}
$$

If we introduce the current density 4-vector

$$
\begin{aligned}
s^\tau(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{g_F(\tau,\vec{\sigma})}}\sum_{i=1}^N Q_i\,\delta(\vec{\sigma}-\vec{\eta}_i(\tau)),\\
s^r(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{g_F(\tau,\vec{\sigma})}}\sum_{i=1}^N Q_i\,\dot{\eta}^r_i(\tau)\,\delta(\vec{\sigma}-\vec{\eta}_i(\tau)),
\end{aligned}
\tag{6.94}
$$

Eqs.(6.89) are replaced by

$$\frac{1}{\sqrt{g_F(\tau,\vec{\sigma})}}\,\frac{\partial}{\partial\sigma^A}\,\Big[\sqrt{g_F(\tau,\vec{\sigma})}\,g_F^{AB}(\tau,\vec{\sigma})\,g_F^{CD}(\tau,\vec{\sigma})F_{BD}\,(\tau,\vec{\sigma})\Big]\,\circeq\,s^C(\tau,\vec{\sigma}). \tag{6.95}$$

From these equations, using the skew-symmetry of F_{AB}, we obtain the continuity equation

$$\frac{1}{\sqrt{g_F(\tau,\vec{\sigma})}}\,\frac{\partial}{\partial\sigma^C}\,\Big[\sqrt{g_F(\tau,\vec{\sigma})}\,s^C(\tau,\vec{\sigma})\Big]\circeq 0. \tag{6.96}$$

This equation can be rewritten in the 3-dimensional form

$$\frac{1}{\sqrt{\gamma_F(\tau,\vec{\sigma})}}\,\frac{\partial}{\partial\tau}\,\Big[\sqrt{\gamma_F(\tau,\vec{\sigma})}\,\overline{\rho}(\tau,\vec{\sigma})\Big]+\frac{1}{\sqrt{\gamma_F(\tau,\vec{\sigma})}}\,\frac{\partial}{\partial\sigma^r}\,\Big[\sqrt{\gamma_F(\tau,\vec{\sigma})}\,\overline{J}^r(\tau,\vec{\sigma})\Big]\circeq 0, \tag{6.97}$$

and implies

$$\frac{d}{d\tau}\,Q_{tot}\circeq 0. \tag{6.98}$$

Let us define the following *generalized non-inertial electric and magnetic fields*

$$\begin{aligned}
\mathcal{E}^s_{(F)}(\tau,\vec{\sigma}) &= -\left[\frac{\sqrt{\gamma_F}}{\sqrt{N_F}}\,\gamma_F^{sr}\,(F_{\tau r}-N_F^v\,F_{vr})\right](\tau,\vec{\sigma})\circeq\pi^s(\tau,\vec{\sigma})),\\
\mathcal{B}^w_{(F)}(\tau,\vec{\sigma}) &= -\frac{1}{2}\,\epsilon^{wsr}\,[N_F\,\sqrt{\gamma_F}\,\gamma_F^{sv}\,\gamma_F^{ru}\,F_{vu}-(N_F^s\,\pi^r-N_F^r\,\pi^s)]\,(\tau,\vec{\sigma}),
\end{aligned} \tag{6.99}$$

where we have introduced the lapse and shift functions $N_F=\sqrt{g_F/\gamma_F}$, $N_F^r=g_{F\,\tau u}\gamma_F^{ur}$. With these new fields the Hamilton equations (6.93) can be written in the form (we use the vector notation as in the 3-dimensional Euclidean case)

$$\begin{aligned}
\nabla\cdot\vec{\mathcal{E}}_{(F)}(\tau,\vec{\sigma}) &= \sqrt{\gamma_F(\tau,\vec{\sigma})}\,\overline{\rho}(\tau,\vec{\sigma}),\\
\frac{\partial\mathcal{E}^r_{(F)}(\tau,\vec{\sigma})}{\partial\tau}-(\nabla\times\vec{\mathcal{B}}_{(F)}(\tau,\vec{\sigma}))^r &= \sqrt{\gamma_F(\tau,\vec{\sigma})}\,\overline{J}^r(\tau,\vec{\sigma}).
\end{aligned} \tag{6.100}$$

With these non-inertial electric and magnetic fields the Hamilton equations look like the usual source dependent Maxwell equations written in a inertial frame.

However it can be useful to introduce the standard definition (see Ref.[108]) of the *inertial electric and magnetic fields*

$$E^r(\tau,\vec{\sigma}) = \eta^r_s\, F_{\tau s}(\tau,\vec{\sigma}), \qquad B^r(\tau,\vec{\sigma}) = -\frac{1}{2}\epsilon^{rsu} F_{rs}(\tau,\vec{\sigma}). \tag{6.101}$$

These fields satisfy the source independent Maxwell equation (existence of the gauge potential) by definition

$$\nabla\times\vec{E}(\tau,\vec{\sigma}) = 0, \qquad \nabla\cdot\vec{B}(\tau,\vec{\sigma}) = 0. \tag{6.102}$$

The source dependent equations for these fields can be found observing that we have

$$\begin{aligned}
\mathcal{E}^s_{(F)}(\tau,\vec{\sigma}) &= \Big[-\frac{\sqrt{\gamma_F}}{\sqrt{N_F}}\,\gamma_F^{sr}\,E^r + \frac{\sqrt{\gamma_F}}{\sqrt{N_F}}\,\gamma_F^{sr}\,(\vec{N}_F\times\vec{B})^r\Big](\tau,\vec{\sigma}),\\
\mathcal{B}^w_{(F)}(\tau,\vec{\sigma}) &= \Big[\frac{1}{2}\,\epsilon^{wsr}\,N_F\,\sqrt{\gamma_F}\,\gamma_F^{sv}\,\gamma_F^{ru}\,\epsilon_{vu\ell}\,B^\ell + (\vec{N}_F\times\vec{E})^w\Big](\tau,\vec{\sigma}),
\end{aligned} \tag{6.103}$$

so that we get

$$\begin{aligned}
\nabla\cdot\vec{E}(\tau,\vec{\sigma}) &= \sqrt{\gamma_F(\tau,\vec{\sigma})}\,\Big[\overline{\rho}(\tau,\vec{\sigma}) - \overline{\rho}_R(\tau,\vec{\sigma})\Big],\\
\frac{\partial E^r(\tau,\vec{\sigma})}{\partial\tau} - (\nabla\times\vec{B}(\tau,\vec{\sigma}))^r &= \sqrt{\gamma_F(\tau,\vec{\sigma})}\,\Big[\overline{J}^r(\tau,\vec{\sigma}) - \overline{J}^r_R(\tau,\vec{\sigma})\Big],
\end{aligned} \tag{6.104}$$

where

$$\begin{aligned}
\overline{\rho}_R(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{\gamma_F(\tau,\vec{\sigma})}}\,\nabla\cdot\Big(\vec{\mathcal{E}}_{(F)}(\tau,\vec{\sigma}) - \vec{E}(\tau,\vec{\sigma})\Big),\\
\overline{J}^r_R(\tau,\vec{\sigma}) &= \frac{1}{\sqrt{\gamma_F(\tau,\vec{\sigma})}}\,\Big[\frac{\partial}{\partial\tau}\,\big(\mathcal{E}^r_{(F)}(\tau,\vec{\sigma}) - E^r(\tau,\vec{\sigma})\big) - \Big(\nabla\times\vec{\mathcal{B}}_{(F)} - \nabla\times\vec{B}\Big)^r\Big]
\end{aligned} \tag{6.105}$$

Due to Eqs(6.103), these charge and current densities are functions only of the metric tensor and of the fields $\vec{E}$, $\vec{B}$.

Also when the gauge fixing constraints (6.77) identify the admissible embedding $z^\mu_F(\tau,\vec{\sigma}) = x^\mu(\tau) + F^\mu(\tau,\vec{\sigma}) = x^\mu(\tau) + \epsilon^\mu_r\,\zeta^r(\tau,\vec{\sigma})$ with $\zeta^r(\tau,\vec{\sigma}) = R^r{}_s(\tau,\sigma)\,\sigma^s$ of Eq.(4.1), whose simultaneity surfaces Σ_τ are space-like hyper-planes with normal $l^\mu = \epsilon^\mu_\tau$, we must use Eqs.(6.105), because the 3-metric g_{rs} of Eqs.(4.5) has a complicate inverse 3-metric.

Instead in Ref.[108] Schiff uses a *non-admissible* $[F(\sigma) = 1,\, \vec{\Omega}(\tau,\sigma) \equiv \vec{\Omega}(\tau)]$ coordinate system of the type (4.1) with $x^\mu(\tau) = u^\mu\,\tau$. In this case, we have

$$N_F(\tau,\vec{\sigma}) = \sqrt{\gamma_F(\tau,\vec{\sigma})} = 1, \quad \gamma^{rs}_F(\tau,\vec{\sigma}) = -\delta^{rs},$$

$$N^r_F(\tau,\vec{\sigma}) = (\vec{\Omega}(\tau)\times\vec{\sigma})^r. \tag{6.106}$$

If we put these expressions in Eqs.(6.103), we find the results of the Appendix A of Ref.[109]

$$\begin{aligned}\vec{\mathcal{E}}_{(F)}(\tau,\vec{\sigma}) &= \vec{E}(\tau,\vec{\sigma}) + (\vec{\Omega}(\tau)\times\vec{\sigma})\times\vec{B}(\tau,\vec{\sigma}),\\ \vec{\mathcal{B}}_{(F)}(\tau,\vec{\sigma}) &= \vec{B} + (\vec{\Omega}(\tau)\times\vec{\sigma})\times\vec{E}(\tau,\vec{\sigma}) + (\vec{\Omega}(\tau)\times\vec{\sigma})\times[(\vec{\Omega}(\tau)\times\vec{\sigma})\times\vec{B}(\tau,\vec{\sigma})],\end{aligned} \tag{6.107}$$

but at the price of a coordinate singularity when $g_{\tau\tau}(\tau,\vec{\sigma})$ vanishes (the horizon problem).

Let us make some remarks

a) Eqs.(6.89) are the generally covariant equations $\nabla_\nu\,F^{\mu\nu} \circeq 0$, suggested by the equivalence principle, in the 3+1 point of view after having taken care of the asymptotic properties at spatial infinity. Eqs. (6.104) and (6.105), with the metric associated to the admissible notion of simultaneity (4.1), should be the starting point for the calculations in the magnetosphere of pulsars, instead of Schiff's equations [111] (6.104) and (6.107), used in Ref.[112], for the case of uniform rotations or of the variants of Refs.[113] (based on Refs.[47]) avoiding the so-called *light cylinder* (the horizon problem) for $\omega\,R = c$, like Eqs.(4.1), but with a bad behavior at spatial infinity.

b) These equations also show that the non-inertial electric and magnetic fields $\vec{\mathcal{E}}_{(F)}$ and $\vec{\mathcal{B}}_{(F)}$ are *not*, in general, *equal* to the fields obtained from the inertial ones $\vec{E}$ and $\vec{B}$ with a Lorentz transformations to the comoving inertial system like it is usually assumed following

Rohrlich [114] and the locality hypothesis [50]. Elsewhere we shall study the system of N charged particles with Grassmann valued electric charges plus the electro-magnetic field in a non-inertial system (till now it was studied only in inertial systems [36]) to understand the energy balance in the case of accelerated charges emitting radiation.

Regarding the *electro-magnetic Dirac observables* on the surface $z^\mu_F(\tau,\vec{\sigma}) = \tilde{x}^\mu(\tau) + F^\mu(\tau,\vec{\sigma})$, let us observe that A_r and π^r admit both a *non-covariant decomposition* [113] in a transverse and a longitudinal part

$$
\begin{aligned}
\pi^r &= \pi^r_\perp + \pi^r_L,\\
&\quad \pi^r_\perp = \epsilon^{rsu}\,\partial_s\,V_u, \qquad \partial_r\,\pi^r_\perp \equiv 0,\\
&\qquad \vec{\partial}\times\vec{\pi}_\perp = \vec{\partial}\times(\vec{\partial}\times\vec{V}) = \vec{\partial}\times\vec{\pi},\\
&\quad \pi^r_L = \tilde{\partial}^r\,V_L, \qquad \tilde{\partial}^r = \delta^{rs}\,\partial_r, \qquad \vec{\partial}\times\vec{\pi}_L \equiv 0,\\
&\quad \partial_r\,\pi^r_L \equiv \triangle V_L = \partial_r\,\pi^r = \Gamma, \qquad \triangle = \partial_r\,\tilde{\partial}^r = \vec{\partial}^2,\\
&\quad V_L = \frac{1}{\triangle}\,\Gamma, \qquad \pi^r_\perp = (\delta^r_s - \tilde{\partial}^r\,\frac{1}{\triangle}\,\partial_s)\,\pi^s,\\
A_r &= A_{\perp r} + A_{L\,r},\\
&\quad A_{\perp r} = \epsilon_{rsu}\,\partial_s\,W_u, \qquad \tilde{\partial}^r\,A_{\perp r} \equiv 0,\\
&\quad \vec{\partial}\times\vec{A}_\perp = \vec{\partial}\times(\vec{\partial}\times\vec{W}) = \vec{\partial}\times\vec{A},\\
&\quad A_{Lr} = \partial_r\,\eta_{em}, \qquad \vec{\partial}\times\vec{A}_L = 0,\\
&\quad \tilde{\partial}^r\,A_{\perp r} \equiv \triangle\,\eta_{em} = \tilde{\partial}^r\,A_r,\\
&\quad \eta_{em} = \frac{1}{\triangle}\,\tilde{\partial}^r\,A_r, \qquad A_{\perp r} = (\delta^s_r - \partial_r\,\frac{1}{\triangle}\,\tilde{\partial}^s)\,A_s,
\end{aligned}
\tag{6.108}
$$

and a *covariant decomposition*

[50] See Ref.[115] for the study of electro-magnetic waves in a standard uniformly rotating frame using the geodesics coordinates of type B quoted in Subsection C of Section I. This study relies on the locality hypothesis and is used to elucidate the phenomenon of helicity-rotation coupling.

$$
\begin{aligned}
\pi^r &= \hat{\pi}^r_\perp + \hat{\pi}^r_L, \qquad \pi_r = g_{F\,rs}\,\pi^s,\\
&\hat{\pi}^r_\perp = (\gamma^{rs}_F - \nabla^r_F\,\frac{1}{\triangle_F}\,\nabla^s_F)\,\pi_s, \qquad \hat{\pi}_{\perp r} = g_{F\,rs}\,\hat{\pi}^s_\perp,\\
&\hat{\pi}^r_L = \nabla^r_F\,\frac{1}{\triangle_F}\,\nabla^s_F\,\pi_s,
\end{aligned}
$$

$$
\begin{aligned}
A_r &= \hat{A}_{\perp r} + \hat{A}_{L\,r}, \qquad A^r = \gamma^{rs}_F\,A_s,\\
&\hat{A}_{\perp r} = (g_{F\,rs} - \nabla_{F\,r}\,\frac{1}{\triangle_F}\,\nabla_{F\,s})\,A^r,\\
&\hat{A}_{L\,r} = \nabla_{F\,r}\,\frac{1}{\triangle_F}\,\nabla_{F\,s}\,A^r.
\end{aligned}
\tag{6.109}
$$

Here ∇^r_F and $\triangle_F = \nabla^r_F\,\nabla_{F\,r} = \frac{1}{\sqrt{\gamma_F(\tau,\vec{\sigma})}}\,\partial_r\left(\sqrt{\gamma_F(\tau,\vec{\sigma})}\,\gamma^{rs}_F(\tau,\vec{\sigma})\,\partial_s\right)$ are the covariant derivative and the Laplace-Beltrami operator associated to $g_{F\,rs}(\tau,\vec{\sigma}|P)$. Since π^r is a vector density, we have $\partial_r\,\pi^r = \nabla_{F\,r}\,\pi^r$.

While with the non-covariant decomposition we can easily find a Shanmugadhasan canonical transformation adapted to the Gauss law constraint with the standard canonically conjugate Dirac observables $\vec{A}_\perp$ and $\vec{\pi}_\perp$ of the radiation gauge, it is not clear whether the covariant decomposition can produce such a canonical basis. In any case, as shown in Ref.[116], the radiation gauge formalism is well defined in both cases if we add suitable gauge fixings.

See Appendix A for a sketch of the derivation of the Sagnac effect from the non-inertial Maxwell equations, following a suggestion of Ref. [58].

With foliations with parallel hyper-planes [$z^\mu_F(\tau,\vec{\sigma}) = x^\mu(\tau) + F^\mu(\tau,\vec{\sigma}) = x^\mu(\tau) + \epsilon^\mu_r\,\zeta^r(\tau,\vec{\sigma})$ with $\zeta^r(\tau,\vec{\sigma}) = R^r{}_s(\tau,\sigma)\,\sigma^s$ of Eq.(4.1)] the constraints (6.77) imply $l^\mu(\tau,\vec{\sigma}) = l^\mu = const.$, i.e. an inertial reference system. As a consequence, the action of the external Lorentz boosts [with the generators (6.73)] on the reduced phase space is broken, because the given conditions $l^\mu = const.$ are compatible only with a subset of the inertial Lorentz frames. Let us remark that the breaking of the canonical action of Lorentz boosts happens also for simultaneity surfaces more general than hyper-planes like those defined by the gauge fixing (6.77), since the form $z^\mu_F(\tau,\vec{\sigma})$ of the embedding is defined in the given reference inertial system.

To recover a good canonical action of the Lorentz group we have to *select a family of admissible embeddings with parallel hyper-planes containing the given embedding with* $l^\mu = const.$ *and all all the embeddings obtained by it by means of Lorentz transformations* (i.e. with $l^\mu = const. \mapsto \Lambda^\mu{}_\nu\, l^\nu = const.$). A similar family of embeddings has to be found also for every type of admissible simultaneity hyper-surfaces identified by the z^μ_F of Eq.(6.77) with a fixed F^μ.

If we define $x^\mu(\tau) = x^\mu(0) + l^\mu\, x_l(\tau) + \epsilon^\mu_r\, x^r_\epsilon(\tau)$ with $x_l(\tau) = \boldsymbol{\epsilon}\, l_\mu\, [x^\mu(\tau) - x^\mu(0)]$ and $x^r_\epsilon(\tau) = -\boldsymbol{\epsilon}\, \epsilon^r_\mu\, [x^\mu(\tau) - x^\mu(0)]$ [ϵ^A_μ are inverse tetrads], the embedding (4.1) is rewritten in the form

$$
\begin{aligned}
z^\mu_F(\tau,\vec{\sigma}) \; &= \; x^\mu(\tau) + F^\mu(\tau,\vec{\sigma}) = x^\mu(0) + l^\mu\, x_l(\tau) + \epsilon^\mu_r\, \xi^r(\tau,\vec{\sigma}),\\
&\qquad \xi^r(\tau,\vec{\sigma}) = x^r_\epsilon(\tau) + \zeta^r(\tau,\vec{\sigma}), \qquad F^\mu(\tau,\vec{\sigma}) = \epsilon^\mu_r\, \zeta^r(\tau,\vec{\sigma}), \quad F^\mu(\tau,\vec{0}) = 0.
\end{aligned}
\tag{6.110}
$$

To describe the above more general family of embeddings, let us modify the Lagrangian $L(\tau)$ of Eqs.(6.71) to $L^{'}(\tau) = L(\tau) - \sqrt{\boldsymbol{\epsilon}\, \dot{X}^2(\tau)}$, by adding the degrees of freedom of a free relativistic particle of unit mass. This amounts to enlarge the phase space by adding the new pairs of canonical variables $X^\mu(\tau)$, $U^\mu(\tau) = \dot{X}^\mu(\tau)/\sqrt{\boldsymbol{\epsilon}\, \dot{X}^2(\tau)}$, $\{X^\mu(\tau), U^\nu(\tau)\} = -\boldsymbol{\epsilon}\, \eta^{\mu\nu}$ restricted by the first class constraint $\chi(\tau) = \boldsymbol{\epsilon}\, U^2(\tau) - 1 \approx 0$, which is added to the Dirac Hamiltonian (6.75), $H_D \mapsto H^{'}_D = H_D + \kappa(\tau)\, \chi(\tau)$ ($\kappa(\tau)$ is a new Dirac multiplier).

Then we replace the embedding (4.1) with the more general embedding [l^μ, $\epsilon^\mu_r \mapsto \hat{U}^\mu(\tau) = U^\mu(\tau)/\sqrt{\boldsymbol{\epsilon}\, U^2(\tau)} \approx U^\mu(\tau)$, $\epsilon^\mu_r(\hat{U}(\tau))$]

$$
\begin{aligned}
z^\mu_{FU}(\tau,\vec{\sigma}) \; &= \; x^\mu(0) + \hat{U}^\mu(\tau)\, x_U(\tau) + \epsilon^\mu_r(\hat{U}(\tau))\, \xi^r_U(\tau,\vec{\sigma}) =\\
&= \; x^\mu_U(\tau) + F^\mu_U(\tau,\vec{\sigma}),\\
&\qquad x^\mu_U(\tau) = x^\mu(0) + \hat{U}^\mu(\tau)\, x_U(\tau) + \epsilon^\mu_r(\hat{U}(\tau))\, x^r_U(\tau),\\
&\qquad \xi^r_U(\tau,\vec{\sigma}) = x^r_U(\tau) + \zeta^r(\tau,\vec{\sigma}), \qquad F^\mu_U(\tau,\vec{\sigma}) = \epsilon^\mu_r(\hat{U}(\tau))\, \zeta^r(\tau,\vec{\sigma}), \quad F^\mu_U(\tau,\vec{0}) = 0,
\end{aligned}
\tag{6.111}
$$

where $\epsilon^\mu_A(\hat{U}) = \Big(\hat{U}^\mu; \epsilon^\mu_r(\hat{U})\Big)$, are the column of the standard Wigner boost sending $\hat{U}^\mu$ at rest.

In this enlarged phase space, studied in detail in Appendix B, we select a special family of 3+1 splittings by means of the gauge fixing constraints [see Eq.(B7)]

$$S(\tau,\vec{\sigma}) \,=\, \hat{U}_\mu(\tau)\,[z^\mu(\tau,\vec{\sigma}) - z^\mu(\tau,\vec{0})] \approx 0,$$

$$\Downarrow$$

$$z^\mu(\tau,\vec{\sigma}) \,\approx\, \theta(\tau)\,\hat{U}^\mu(\tau) + \epsilon^\mu_r(\hat{U}(\tau))\,\mathcal{A}^r(\tau,\vec{\sigma}). \qquad (6.112)$$

The admissible foliations of this family have space-like hyper-planes, orthogonal to the arbitrary unit vector $\hat{U}^\mu(\tau)$, as simultaneity and Cauchy leaves. The centroid $z^\mu(\tau,\vec{0}) = \theta(\tau)\,\hat{U}^\mu(\tau) + \epsilon^\mu_r(\hat{U}(\tau))\,\mathcal{A}^r(\tau,\vec{0})$, origin of the 3-coordinates, describes an arbitrary non-inertial time-like observer and on the hyper-planes there is an arbitrary admissible rotating frame determined by the functions $\mathcal{A}^r(\tau,\vec{\sigma})$. As shown by Eqs. (B8)-(B10), if we make the decomposition $\rho^\mu(\tau,\vec{\sigma}) = \boldsymbol{\epsilon}\,\Big([M_U(\tau) + \tilde{\rho}_U(\tau,\vec{\sigma})]\,\hat{U}^\mu(\tau) - \epsilon^\mu_r(\hat{U}(\tau))\,\rho_{Ur}(\tau,\vec{\sigma})\Big)$ [51], of the momentum of Eq.(6.72), then the gauge fixing (6.112) together with the constraint $\hat{U}^\mu(\tau)\,[\mathcal{H}_\mu(\tau,\vec{\sigma}) - \int d^3\sigma_1\,\mathcal{H}_\mu(\tau,\vec{\sigma}_1)] \approx 0$ [52] form a pair of second class constraints, which can be eliminated by going to Dirac brackets.

As shown in Appendix B a canonical basis for this new reduced phase space is $\theta(\tau)$, $M_U(\tau)$, $\mathcal{A}^r(\tau,\vec{\sigma})$, $\rho_{Ur}(\tau,\vec{\sigma})$, $\tilde{X}^\mu(\tau)$, $U^\mu(\tau)$ plus the electro-magnetic canonical variables. Now we have $l^\mu(\tau,\vec{\sigma}) \equiv \hat{U}^\mu(\tau)$, $z^\mu_r(\tau,\vec{\sigma}) \equiv \epsilon^\mu_s(\hat{U}(\tau))\,\frac{\partial \mathcal{A}^s(\tau,\vec{\sigma})}{\partial \sigma^r}$, $g_{rs}(\tau,\vec{\sigma}) \equiv -\boldsymbol{\epsilon}\,\sum_u\,\frac{\partial \mathcal{A}^u(\tau,\vec{\sigma})}{\partial \sigma^r}\,\frac{\partial \mathcal{A}^u(\tau,\vec{\sigma})}{\partial \sigma^s}$ and $\boldsymbol{\epsilon}\,\tilde{\rho}_U(\tau,\vec{\sigma}) \equiv T_{\tau\tau}(\tau,\vec{\sigma}) - \int d^3\sigma_1\,T_{\tau\tau}(\tau,\vec{\sigma}_1)$. The remaining first class constraints are $\chi(\tau) \approx 0$, $H_\perp = M_U(\tau) - \int d^3\sigma\,T_{\tau\tau}(\tau,\vec{\sigma}) \approx 0$, $\mathcal{H}_r(\tau,\vec{\sigma}) = \frac{\partial \mathcal{A}^s(\tau,\vec{\sigma})}{\partial \sigma^r}\,\rho_{Us}(\tau,\vec{\sigma}) - \boldsymbol{\epsilon}\,T_{\tau r}(\tau,\vec{\sigma}) \approx 0$ with the components of the energy-momentum tensor given in Eqs.(6.74). The canonical 4-coordinate $\tilde{X}^\mu(\tau) = X^\mu(\tau) + W^\mu(\tau)$ is not a 4-vector [53]: $\tilde{X}^\mu$ plays the role of the decoupled 4-center of mass of the accelerated isolated system. See Appendix B for the form of the Poincare' generators. *The resulting breaking of the canonical action of the Lorentz boosts is restricted*

---

[51] With $\rho_U(\tau,\vec{\sigma}) = \hat{U}^\mu(\tau)\,\rho_\mu(\tau,\vec{\sigma})$, $\rho_{Ur}(\tau,\vec{\sigma}) = \epsilon^\mu_r(\hat{U}(\tau))\,\rho_\mu(\tau,\vec{\sigma})$, $M_U(\tau) = \int d^3\sigma\,\rho_U(\tau,\vec{\sigma})$, $\tilde{\rho}_U(\tau,\vec{\sigma}) = \rho_U(\tau,\vec{\sigma}) - M_U(\tau)$.

[52] Determining $\tilde{\rho}_U(\tau,\vec{\sigma}) \approx \hat{U}^\mu(\tau)\,[l_\mu(\tau,\vec{\sigma})\,T_{\tau\tau}(\tau,\vec{\sigma}) + z_{r\mu}(\tau,\vec{\sigma})\,\gamma^{rs}(\tau,\vec{\sigma})\,T_{\tau s}(\tau,\vec{\sigma}) - \int d^3\sigma_1 (l_\mu\,T_{\tau\tau} + z_{r\mu}\,\gamma^{rs}\,T_{\tau s})(\tau,\vec{\sigma}_1)]$.

[53] Like it happens with the decoupled external 4-center of mass $\tilde{x}^\mu(\tau)$ in the rest-frame instant form in inertial frames.

to the gauge variable $\tilde{X}^\mu$. As in the rest-frame instant form we can make a canonical transformation from the canonical basis $\tilde{X}^\mu$, U^μ to one spanned by $\hat{U}_\mu\, \tilde{X}^\mu = \hat{U}_\mu\, X^\mu$ [since $\hat{U}_\mu\, W^\mu = 0$], $\sqrt{\boldsymbol{\epsilon}\, U^2} \approx 1$, $\vec{z} = \sqrt{\boldsymbol{\epsilon}\, U^2}\, [\vec{\tilde{X}} - \tilde{X}^o\, \vec{U}/U^o] \approx \vec{\tilde{X}} - \tilde{X}^o\, \vec{U}/U^o$, $\vec{k} = \vec{U}/\sqrt{\boldsymbol{\epsilon}\, U^2}\, \vec{U} \approx \vec{\hat{U}}$, with $\vec{z}$ and $\vec{k}$ non-evolving Jacobi initial data.

If we want to recover the embedding (6.111), we have to add the gauge fixings $\theta(\tau) - x_U(\tau) - \hat{U}_\mu(\tau)\, x^\mu(0) \approx 0$, $\mathcal{A}^r(\tau, \vec{\sigma}) - \xi^r_U(\tau, \vec{\sigma}) - \epsilon^r_\mu(\hat{U}(\tau))\, x^\mu(0) \approx 0$ with $x_U(\tau)$, $\xi^r_U(\tau, \vec{\sigma}) = x^r_U(\tau) + \zeta^r(\tau, \vec{\sigma})$ [$\zeta^r(\tau, \vec{0}) = 0$] given ($U$-independent) functions. This implies $z^\mu(\tau, \vec{\sigma}) \approx z^\mu_{FU}(\tau, \vec{\sigma})$ and $z^\mu(\tau, \vec{0}) = x^\mu_U(\tau) = x^\mu(0) + \hat{U}^\mu(\tau)\, x_U(\tau) + \epsilon^\mu_r(\hat{U}(\tau))\, x^r_U(\tau)$, i.e. a family of admissible 3+1 splittings whose whose simultaneity leaves are hyper-planes orthogonal to $\hat{U}^\mu(\tau)$ and with rotating 3-coordinates determined by the functions $\zeta^r(\tau, \vec{\sigma})$ [for instance the admissible ones of Eqs.(4.1)]. By going to new Dirac brackets we get a new reduced phase space spanned by $\tilde{X}^\mu(\tau)$, $U^\mu(\tau)$ and the electro-magnetic canonical variables.

The natural gauge fixing to the constraint $\chi(\tau) = \boldsymbol{\epsilon}\, U^2(\tau) - 1 \approx 0$ is $\hat{U}_\mu(\tau)\, \tilde{X}^\mu(\tau) - \boldsymbol{\epsilon}\, \theta(\tau) \approx 0$: it replaces the gauge fixing $\tau - u \cdot \tilde{x} \approx 0$ of the rest-frame instant form. After this gauge fixing we have $\tilde{X}^\mu(\tau) = X^\mu(\tau) + W^\mu(\tau) = z^\mu(\tau, \vec{\sigma}_{\tilde{X}}(\tau))$ and $X^\mu(\tau) = z^\mu(\tau, \vec{\sigma}_X(\tau))$ for some $\vec{\sigma}_{\tilde{X}}(\tau)$ and $\vec{\sigma}_X(\tau)$.

If finally we want to recover the embedding (4.1), we must add by hand three more first class constraints, the independent ones in $\hat{U}^\mu(\tau) \approx l^\mu = \hat{U}^\mu(\vec{k}) = const.$, which determine $\vec{k}$. As gauge fixings to these three extra constraints it is natural to choose $\vec{z} \approx 0$. In this way $\vec{\sigma}_{\tilde{X}}(\tau)$ is determined.

Therefore the description of non-inertial isolated systems follows a pattern similar to that needed for their description in the inertial system of the rest-frame instant form. There is a decoupled non-covariant canonical variable, needed for the canonical implementation of the external Lorentz transformations. However now it does not carry the conserved 4-momentum of the isolated system, which is associated to the centroid describing a non-inertial observer.

The formalism developed in this Subsection and in Appendix B will be needed to implement the program of quantization of the electro-magnetic field in non-inertial systems along the lines under study for relativistic particles in Ref.[117].

Finally see Refs.[58, 70, 115, 118] for what is known on the open problem of the constitutive equations for electrodynamics in material media in non-inertial systems.

VII. CONCLUSIONS.

In the Introduction we have reviewed many old and new physical problems in special relativity, which are naturally formulated in accelerated (in particular rotating) frames. We have stressed that they present pathologies (coordinate singularities) originating from the absence of an admissible notion of simultaneity (i.e. of a frame-dependent rule for the synchronization of distant clocks to the reference clock of a given observer). Then we have analyzed in detail which are the conditions to be imposed on coordinate transformations, starting from the standard Cartesian coordinates of Minkowski space-time, so that the new *equal time surfaces* (instantaneous 3-spaces) are the space-like leaves of the foliation associated to an admissible 3+1 splitting of Minkowski space-time. Einstein's convention is a very special case and corresponds to the space-like hyper-planes orthogonal to the world-line of an inertial observer. More in general the leaves of a foliation will have both a linear acceleration, describing how they are packed, and a parametrization with differentially rotating 3-coordinate systems. It turns out that, while there is no restriction on linear accelerations, on the contrary angular velocities and rotational accelerations cannot be given arbitrarily, but must be suitably restricted. In particular rigid rotations are not allowed.

In this paper it is pointed out that it is convenient to characterize the admissible 3+1 splittings of Minkowski space-time with *intrinsic Lorentz-invariant radar 4-coordinates* $\sigma^A = (\tau, \vec{\sigma})$ [τ labels the leaves and $\vec{\sigma} = (\sigma^r)$ are curvilinear 3-coordinates on the leaf Σ_τ with respect to an arbitrary centroid $x^\mu(\tau)$], which parametrize the *embedding* $x^\mu = z^\mu(\tau, \vec{\sigma})$ of the leaves in Minkowski space-time. We have explicitly built a family of such admissible radar coordinates implementing the *locality hypothesis*. A sub-family of these 4-coordinates corresponds to foliations of Minkowski space-time with parallel (but in general not equi-spaced) hyper-planes endowed with suitable differentially rotating 3-coordinates. In particular all these admissible foliations of Minkowski space-time are associated to arbitrary accelerated time-like observers, whose world-line $x^\mu(\tau)$ is the centroid origin of the 3-coordinates. The *equal time ($\tau = const.$) surfaces* (instantaneous 3-spaces) are *not orthogonal* to the world-line of the observer and their associated notion of simultaneity corresponds to a modification of Einstein's convention for the synchronization of clocks. Moreover we have shown that given an admissible foliation with simultaneity surfaces, there are two associated congruences of time-like (in general non-inertial) observers. One non-rotating determined by the normals to the simultaneity surfaces and one rotating (non-surface-forming) determined by the τ-time

derivative of the embedding $z^\mu(\tau,\vec{\sigma})$.

All the admissible notions of simultaneity are *gauge equivalent* for the description of an isolated system, when it admits a formulation in terms of a parametrized Minkowski theory. In this case the Lagrangian density depends on the embedding $z^\mu(\tau,\vec{\sigma})$ besides on the variables of the system and there is a special type of general covariance peculiar to special relativity. As a consequence the embeddings are *gauge variables* and the transition from a 3+1 splitting to another one (i.e. the change of the notion of simultaneity) is a gauge transformation like the change of the 3-coordinates $\vec{\sigma}$. Therefore parametrized Minkowski theories, which have local invariance (second Noether theorem) under the sub-group of frame preserving diffeomorphisms $\tau \mapsto f(\tau,\vec{\sigma})$, $\vec{\sigma} \mapsto \vec{g}(\vec{\sigma})$, are generally covariant theories like general relativity, where the embedding $z^\mu(\tau,\vec{\sigma})$ are replaced by the 4-metric and the Hilbert action has local invariance under the full group of diffeomorphisms. In canonical metric gravity the change of the foliation (i.e. of the notion of simultaneity) is a Hamiltonian gauge transformation generated by the super-Hamiltonian constraint [88], while the super-momentum constraints are the generators of the change of the 3-coordinates adapted to the foliation. Parametrized Minkowski theories are a non-trivial genuine (i.e. not artificial) example which validates Kretschmann's rejection of Einstein's argument that only general relativity has a general covariance group [119].

Let us emphasize that the real novelty of canonical gravity is that each solution of Einstein's equations, i.e. of the Hamilton equations in a completely fixed gauge, with given boundary conditions and allowed initial data also determines the extrinsic curvature of the Cauchy surfaces, which are then found solving an inverse problem. As a consequence, *the dynamics of the gravitational field determines the admissible notions of simultaneity in general relativity* [**?**]: they are much less than the admissible ones of special relativity, because in absence of matter, as said in the second paper of Ref.[96], they must have the leaves 3-conformally flat.

Let us remark that there is a distinction between notions and/or results independent from the choice of the notion of simultaneity (i.e. they are observer-independent, like the statement that two events A and B have a space-like separation) and those which are frame-dependent (like the solutions of the equations of motion after a well defined choice of the Cauchy-simultaneity surface). However, like in general relativity [8], the definition of an extended laboratory, with its instruments and its standard units, corresponds to a

well defined choice of the notion of simultaneity, of an associated notion of spatial distance and of an associated set of adapted 4-coordinates (or of their intrinsic variant, the radar coordinates). Therefore a laboratory will always give a frame-dependent description of physical systems. A change of the notion of simultaneity and of the 4-coordinates will only produce *inertial* effects in the description of the motion of particles and fields.

The 3+1 point of view, in which the simultaneity surfaces are also Cauchy surfaces for the equation of motion of isolated systems, has been contrasted in this paper with the 1+3 point of view of either an accelerated observer or a rotating congruence of observers like the one determined by the τ-time derivative of an embedding $z^\mu(\tau, \vec{\sigma})$. While, after having endowed the observer with a tetrad (whose space axes are arbitrarily chosen), the 1+3 point of view is the only one allowing to define the tetradic (coordinate-independent but tetrad-dependent) components of the fields (like the electro-magnetic field) to be measured locally by the observer, only in the 3+1 point of view we have a well posed Cauchy problem and a control on the predictability of the theory.

We have analyzed the gauge nature (frame-dependence) of the notions of one-way velocity of light and spatial distance and compared the results of the 3+1 point of view (admissible global notions of simultaneity and instantaneous 3-space) with those of the 1+3 one (only approximate non-global synchronizability of clocks and non-existence of an instantaneous 3-space, locally replaced with the 3-space of the vectors orthogonal to the observer world-line).

Then we have applied the formalism of the admissible 4-coordinate transformations to various problems.

A) We have delineated a method for building a grid of radar 4-coordinates after having assigned an admissible modification of Einstein convention plus a convention on how to build fixed-time 3-coordinates. This method could be used by a set of spacecrafts or satellites like in the GPS setting.

B) We have given the 3+1 point of view on the rotating disk and the Sagnac effect.

C) We have evaluated the correction of order c^{-3} to the one-way time transfer between an Earth station and a satellite due to the rotation of the Earth, after having established a grid of radar coordinates like in A) and we have given a re-interpretation of the ACES mission

as the determination of the deviation from Einstein's convention of the chosen notion of simultaneity. Similar calculations could be done for LISA and VLBI.

D) We have studied the description of the electro-magnetic field with a parametrized Minkowski theory and analyzed in detail the restriction to an arbitrary notion of simultaneity. As a consequence we have determined the most general form of Maxwell equations in a non-inertial system and studied in detail the case of foliations with parallel hyper-planes and rotating 3-coordinates. This will be useful in the study of the magneto-sphere of pulsars and of the energy balance for the radiation emitted by accelerated charges.

The technology developed in D) will be needed for the study of a new method of quantization of relativistic particles and of the electro-magnetic field in non-inertial frames [117].

APPENDIX A: THE SAGNAC EFFECT FROM NON-INERTIAL MAXWELL EQUATIONS.

Let us now sketch how it is possible to derive the Sagnac effect from Maxwell equations in non-inertial system as suggested in Ref.[58].

To find the bridge between the geometric derivation of the Sagnac effect and the non-inertial equations of motion of the electro-magnetic field, we can use the *eikonal approximation*. To do this we specify a embedding $z^\mu_F(\tau,\vec{\sigma})$ of the form (6.110), that is such that the hyper-surfaces are parallel hyper-planes with constant normal l^μ.

It is convenient to use gauge fixed vector potentials $A_B(\tau,\vec{\sigma})$ satisfying the conditions

$$A_N(\tau,\vec{\sigma}) = \left[\frac{1}{N_F}\,(A_\tau - N^r_F\,A_r)\right](\tau,\vec{\sigma}) = 0,$$

$$\frac{1}{\sqrt{\gamma_F(\tau,\vec{\sigma})}}\frac{\partial}{\partial\sigma^r}\,\Big(\sqrt{\gamma_F(\tau,\vec{\sigma})}\,\gamma^{rs}_F(\tau,\vec{\sigma})\,A_s(\tau,\vec{\sigma})\Big) = 0. \tag{A1}$$

These conditions correspond to a radiation gauge for the inertial observers with coordinates τ, $\xi^r(\tau,\vec{\sigma}) = R^r{}_s(\tau,\sigma)\,\sigma^s$. Using the notations of Eq.(6.110), they imply

$$\frac{1}{\sqrt{g_F(\tau,\vec{\sigma})}}\frac{\partial}{\partial\sigma^A}\,\Big(\sqrt{g_F(\tau,\vec{\sigma})}\,g^{AB}_F(\tau,\vec{\sigma})\,A_B(\tau,\vec{\sigma})\Big) = 0. \tag{A2}$$

Then we make the following *ansatz* for the potential

$$A_B(\tau,\vec{\sigma}) = \frac{1}{\omega^2}\,\mathcal{A}_B(\tau,\vec{\sigma})\,\exp[\,i\,\omega\,\Phi(\tau,\vec{\sigma})], \tag{A3}$$

where ω is a frequency. We assume the validity of the following conditions (*eikonal approximation*)

$$\omega >> 1, \qquad \left|\frac{\partial\mathcal{A}_B(\tau,\vec{\sigma})}{\partial\sigma^C}\right| << 1. \tag{A4}$$

At order $1/\omega$ the equations of motion (6.89) give

$$\left[\mathcal{A}_D\;g^{AB}_F\;\frac{\partial\Phi}{\partial\sigma^A}\,\frac{\partial\Phi}{\partial\sigma^B}\;-\frac{\partial\Phi}{\partial\sigma^D}\,\mathcal{A}_B\;g^{AB}_F\;\frac{\partial\Phi}{\partial\sigma^A}\right](\tau,\vec{\sigma}) + \mathcal{O}(1/\omega) = 0. \tag{A5}$$

The condition (A2) gives at the same order

$$\left[\mathcal{A}_B\;g^{BA}_F\;\frac{\partial\Phi}{\partial\sigma^A}\right](\tau,\vec{\sigma}) + \mathcal{O}(1/\omega) = 0. \tag{A6}$$

Therefore we get the *eikonal equation*

$$g_F^{AB}(\tau,\vec{\sigma})\,\frac{\partial\Phi}{\partial\sigma^A}(\tau,\vec{\sigma})\,\frac{\partial\Phi}{\partial\sigma^B}(\tau,\vec{\sigma})=0. \tag{A7}$$

Let us make the *ansatz* that there is a solution of the type (this is the weak point of the derivation, because strictly speaking this solution requires a static metric; one should show that the deviations from the static case are negligible!)

$$\Phi(\tau,\vec{\sigma})=\tau+\Psi(\vec{\sigma}). \tag{A8}$$

We want to evaluate the infinitesimal variation of the Ψ along a infinitesimal 3-dimensional displacement tangent to a curve $\sigma^r(\lambda)$. Namely we want to evaluate

$$d\Psi(\vec{\sigma}(\lambda))=\frac{\partial\Psi}{\partial\sigma^r}(\vec{\sigma}(\lambda))\cdot\frac{d\sigma^r(\lambda)}{d\lambda}\,d\lambda. \tag{A9}$$

To do this, we transform the 3-dimensional curve $\sigma^r(\lambda)$ in Minkowski space-time in a world-line by introducing a $\tau(\lambda)$ such that

$$\Phi(\tau(\lambda),\vec{\sigma}(\lambda))=const., \tag{A10}$$

so that we get

$$\frac{\partial\Phi}{\partial\sigma^B}(\tau(\lambda),\vec{\sigma}(\lambda))\cdot\frac{d\sigma^B(\lambda)}{d\lambda}=0. \tag{A11}$$

From the eikonal equation (A7) we obtain

$$\alpha(\lambda)\,\frac{d\sigma^B(\lambda)}{d\lambda}=\left[g_F^{BA}\,\frac{\partial\Phi}{\partial\sigma^A}\right](\tau(\lambda),\vec{\sigma}(\lambda)), \tag{A12}$$

where $\alpha(\lambda)$ is a multiplier depending of the choice of the affine parameter λ. Then we also have

$$\frac{d\sigma^A(\lambda)}{d\lambda}\cdot g_F^{AB}(\tau(\lambda),\vec{\sigma}(\lambda))\cdot\frac{d\sigma^B(\lambda)}{d\lambda}=0. \tag{A13}$$

Therefore $\sigma^A(\lambda)$ is a null curve. From Eq.(A12) and using the (A8), we obtain

$$\begin{aligned}\alpha(\lambda)\,\frac{d\tau(\lambda)}{d\lambda}&=\left[g_F^{\tau r}\,\frac{\partial\Psi}{\partial\sigma^r}+g_F^{\tau\tau}\right](\tau(\lambda),\vec{\sigma}(\lambda)),\\ \alpha(\lambda)\,\frac{d\sigma^r(\lambda)}{d\lambda}&=\left[g_F^{rs}\,\frac{\partial\Psi}{\partial\sigma^s}+g_F^{\tau r}\right](\tau(\lambda),\vec{\sigma}(\lambda)).\end{aligned} \tag{A14}$$

Using the second of these equations we obtain

$$\alpha(\lambda)\,\frac{\partial\Psi}{\partial\sigma^r}(\vec{\sigma}(\lambda))\cdot\frac{d\sigma^r(\lambda)}{d\lambda}=\left[\frac{\partial\Psi}{\partial\sigma^r}\,g_F^{rs}\,\frac{\partial\Psi}{\partial\sigma^s}+\frac{\partial\Psi}{\partial\sigma^r}g_F^{\tau r}\right](\tau(\lambda),\vec{\sigma}(\lambda)).\tag{A15}$$

Summing the first of Eqs.(A14) with Eq. (A15) and using Eqs.(A7) and (A8), we obtain

$$\frac{\partial\Psi}{\partial\sigma^r}(\vec{\sigma}(\lambda))\cdot\frac{d\sigma^r(\lambda)}{d\lambda}=-\frac{d\tau(\lambda)}{d\lambda}.\tag{A16}$$

We can think a ray of light constrained to follow a curve $\vec{\sigma}(\lambda)$ as a sequence of wave plane solutions of the type (A3) covering infinitesimal distances $(d\sigma^r(\lambda)/d\lambda)d\lambda$. Then the phase shift accumulate by this ray of light along a finite length on the curve is

$$\Delta\Psi=-\int d\lambda\,\frac{d\tau(\lambda)}{d\lambda},\tag{A17}$$

where $\tau(\lambda),\vec{\sigma}(\lambda)$ is a null curve.

This justifies the geometrical calculus of Subsection C of Section VI. In that case the 3-dimensional curve $\vec{\sigma}(\varphi)$ was the circle on the hyperplane and the $\tau(\varphi)$ was built imposing Eqs.(A13) [see Eq.(6.26) for $\varphi(\tau)$] obtaining so the two solutions corresponding to the two directions.

APPENDIX B: A FAMILY OF FOLIATIONS WITH HYPER-PLANES CLOSED UNDER LORENTZ TRANSFORMATIONS.

In this Appendix we study a family of admissible embeddings with parallel hyper-planes closed under the action of the Lorentz transformations of the inertial system, namely such that a Lorentz transformation maps one member of the family onto another of its members.

Let us consider the U-dependent family of embeddings of Eq.(6.111)

$$\begin{aligned} z^\mu_{FU}(\tau,\vec{\sigma}) &= x^\mu(0) + \hat{U}^\mu(\tau)\, x_U(\tau) + \epsilon^\mu_r(\hat{U}(\tau))\, \xi^r_U(\tau,\vec{\sigma}) = \\ &= x^\mu_U(\tau) + F^\mu_U(\tau,\vec{\sigma}), \qquad F^\mu_U(\tau,\vec{0}) = 0, \end{aligned}$$

$$\begin{aligned} & x^\mu_U(\tau) = x^\mu(0) + \hat{U}^\mu(\tau)\, x_U(\tau) + \epsilon^\mu_r(\hat{U}(\tau))\, x^r_U(\tau), \\ & \xi^r_U(\tau,\vec{\sigma}) = x^r_U(\tau) + \zeta^r(\tau,\vec{\sigma}), \qquad F^\mu_U(\tau,\vec{\sigma}) = \epsilon^\mu_r(\hat{U}(\tau))\, \zeta^r(\tau,\vec{\sigma}), \end{aligned} \tag{B1}$$

where $\hat{U}^\mu(\tau)$ is the unit normal to the hyper-surface and $\hat{U}^\mu$, $\epsilon^\mu_r(\hat{U})$ are the columns of the standard Wigner boost sending $\hat{U}^\mu$ at rest. As a consequence the hyper-sufaces of this foliations are parallel hyper-planes. The $\epsilon^\mu_a(\hat{U})$ are a triad of space-like four-vector such that [89]

$$\begin{aligned} & \hat{U}_\mu\, \epsilon^\mu_a(\hat{U}) = 0, \quad \epsilon^\mu_a(\hat{U})\epsilon_{\mu\, b}(\hat{U}) = \eta_{ab}, \quad \hat{U}_\mu\, \frac{\partial \epsilon^\lambda_a(\hat{U})}{\partial \hat{U}_\mu} = 0, \\ & \epsilon^\mu_a(\Lambda\, \hat{U}) = \Lambda^\mu{}_\nu\, \epsilon^\nu_b(\hat{U}) R_{ba}(\Lambda, \hat{U}), \end{aligned} \tag{B2}$$

where $R_{ba}(\Lambda, \hat{U})$ is the Wigner rotation associed to the Lorentz transformation Λ by the standard Wigner boost $L(\hat{U}, \hat{U}_o)$, such that $\hat{U}^\mu = L^\mu_\nu(\hat{U}, \hat{U}_o)\, \hat{U}^\nu_o$ ($\hat{U}^\nu_o = (1,0,0,0)$). By definition we have $[L(\hat{U}, \hat{U}_o)\, \Lambda\, L^{-1}(\hat{U}, \hat{U}_o)]^i{}_j = R_{a=i,b=j}(\Lambda, \hat{U})$, $[L(\hat{U}, \hat{U}_o)\, \Lambda\, L^{-1}(\hat{U}, \hat{U}_o)]^o{}_o = 1$, $[L(\hat{U}, \hat{U}_o)\, \Lambda\, L^{-1}(\hat{U}, \hat{U}_o)]^i{}_o = [L(\hat{U}, \hat{U}_o)\, \Lambda\, L^{-1}(\hat{U}, \hat{U}_o)]^o{}_j = 0$.

When we add the free relativistic particle $X^\mu(\tau)$ of unit mass to the Lagrangian (6.71), its conjugate momentum $U^\mu(\tau)$ ($\{X^\mu(\tau), U_\nu(\tau)\} = -\eta^\mu_\nu$) realizes the parameter of the family as a canonical variable, which satisfies the extra first class constraint

$$\chi(\tau) = \boldsymbol{\epsilon}\, U^2(\tau) - 1 \approx 0, \quad \Rightarrow \quad \hat{U}^\mu(\tau) = \frac{U^\mu(\tau)}{\sqrt{\boldsymbol{\epsilon}\, U^2(\tau)}} \approx U^\mu(\tau). \tag{B3}$$

The new Dirac hamiltonian (see Eq.(6.75); we momentarily ignore the electro-magnetic constraints) is

$$H_D(\tau) = \int d^3\sigma \left[\tilde{\lambda}_\perp(\tau,\vec{\sigma})\mathcal{H}_\perp(\tau,\vec{\sigma}) + \tilde{\lambda}^r(\tau,\vec{\sigma})\mathcal{H}_r(\tau,\vec{\sigma})\right] + \kappa(\tau)\,\chi(\tau). \tag{B4}$$

The canonical generators (6.73) of the Poincaré group are replaced by

$$\begin{aligned} P^\mu_s(\tau) &= U^\mu(\tau) + \int d^3\sigma\, \rho^\mu(\tau,\vec{\sigma}), \\ J^{\mu\nu}_s(\tau) &= X^\mu(\tau)U^\nu(\tau) - X^\nu(\tau)U^\mu(\tau) + \int d^3\sigma\, [z^\mu(\tau,\vec{\sigma})\rho^\nu(\tau,\vec{\sigma}) - z^\nu(\tau,\vec{\sigma})\rho^\mu(\tau,\vec{\sigma})]. \end{aligned} \tag{B5}$$

To identify the embeddings (B1) we cannot use the gauge fixings (6.77) implying $z^\mu(\tau,\vec{\sigma}) \approx z^\mu_F(\tau,\vec{\sigma}) = x^\mu(\tau) + F^\mu(\tau,\vec{\sigma})$. Instead we have to introduce the gauge fixing

$$S(\tau,\vec{\sigma}) = \hat{U}^\mu(\tau)\ [z_\mu(\tau,\vec{\sigma}) - z_\mu(\tau,0)] \approx 0, \tag{B6}$$

implying

$$\begin{aligned} z^\mu(\tau,\vec{\sigma}) &\approx \theta(\tau)\,\hat{U}^\mu(\tau) + \epsilon^\mu_r(\hat{U}(\tau))\,\mathcal{A}^r(\tau,\vec{\sigma}) = \\ &= z^\mu(\tau,\vec{0}) + \epsilon^\mu_r(U(\tau))\left[\mathcal{A}^r(\tau,\vec{\sigma}) - \mathcal{A}^r(\tau,\vec{0})\right], \\ &\theta(\tau) = \boldsymbol{\epsilon}\,\hat{U}^\mu(\tau)\,z_\mu(\tau,\vec{0}), \qquad \mathcal{A}^r(\tau,\vec{\sigma}) = -\boldsymbol{\epsilon}\,\epsilon^r_\mu(\hat{U}(\tau))\,z^\mu(\tau,\vec{\sigma}), \\ &z^\mu(\tau,\vec{0}) = \theta(\tau)\,\hat{U}^\mu(\tau) + \epsilon^\mu_r(\hat{U}(\tau))\,\mathcal{A}^r(\tau,\vec{0}), \\ &z^\mu_r(\tau,\vec{\sigma}) \approx \epsilon^\mu_s(\hat{U}(\tau))\,\frac{\partial \mathcal{A}^s(\tau,\vec{\sigma})}{\partial\sigma^r}, \quad \Rightarrow \quad l^\mu(\tau,\vec{\sigma}) \approx \hat{U}^\mu(\tau), \\ \Rightarrow \qquad &g_{rs}(\tau,\vec{\sigma}) \approx -\boldsymbol{\epsilon}\sum_u \frac{\partial \mathcal{A}^u(\tau,\vec{\sigma})}{\partial\sigma^s}\,\frac{\partial \mathcal{A}^u(\tau,\vec{\sigma})}{\partial\sigma^s}. \end{aligned} \tag{B7}$$

Therefore the gauge fixing (B6) implies that the simultaneity surfaces Σ_τ are hyper-planes orthogonal to the arbitrary time-like unit vector $\hat{U}^\mu(\tau)$, which is a constant of the motion since Eq.(B4) implies $\frac{d\hat{U}^\mu(\tau)}{d\tau} \circeq 0$.

The time preservation of the gauge fixing (B6) implies

$$\frac{d}{d\tau}\, S(\tau, \vec{\sigma}) \approx 0 \,\Rightarrow\, \tilde{\lambda}_{\perp}(\tau, \vec{\sigma}) - \tilde{\lambda}_{\perp}(\tau, 0) \approx 0 \,\Rightarrow\, \tilde{\lambda}_{\perp}(\tau, \vec{\sigma}) \approx \mu(\tau), \tag{B8}$$

and then in the reduced theory we have the Dirac Hamiltonian (still ignoring the electro-magnetic constraints)

$$H_D(\tau) = \mu(\tau)\mathcal{H}_{\perp}(\tau) + \int d^3\sigma\, \tilde{\lambda}^r(\tau, \vec{\sigma})\mathcal{H}_r(\tau, \vec{\sigma}) + \kappa(\tau)\,\chi(\tau), \tag{B9}$$

where

$$H_{\perp}(\tau) = \int d^3\sigma\, \mathcal{H}_{\perp}(\tau, \vec{\sigma}) \approx 0, \tag{B10}$$

and [see Eqs.(6.74)]

$$\begin{aligned}
\mathcal{H}_{\perp}(\tau, \vec{\sigma}) \;&=\; l^{\mu}(\tau, \vec{\sigma})\, \mathcal{H}_{\mu}(\tau, \vec{\sigma}) \approx \mathcal{H}_U(\tau, \vec{\sigma}) = \hat{U}^{\mu}(\tau)\, \mathcal{H}_{\mu}(\tau, \vec{\sigma}) \approx \\
&\approx\; \rho_U(\tau, \vec{\sigma}) - \boldsymbol{\epsilon}\, T_{\tau\tau}(z^{\mu}(\tau, \vec{\sigma}), \mathcal{I}) \approx 0,
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_r(\tau, \vec{\sigma}) \;&=\; z^{\mu}_r(\tau, \vec{\sigma})\, \mathcal{H}_{\mu}(\tau, \vec{\sigma}) = z^{\mu}_r(\tau, \vec{\sigma})\rho_{\mu}(\tau, \vec{\sigma}) - \boldsymbol{\epsilon}\, T_{\tau r}(z^{\mu}(\tau, \vec{\sigma}), \mathcal{I}) \approx \\
&\approx\; \epsilon^{\mu}_s(\hat{U}(\tau))\, \frac{\partial \mathcal{A}^s(\tau, \vec{\sigma})}{\partial \sigma^r}\, \mathcal{H}_{\mu}(\tau, \vec{\sigma}) = \frac{\partial \mathcal{A}^s(\tau, \vec{\sigma})}{\partial \sigma^r}\, \mathcal{H}_{Us}(\tau, \vec{\sigma}) = \\
&=\; \frac{\partial \mathcal{A}^s(\tau, \vec{\sigma})}{\partial \sigma^r}\, \rho_{Us}(\tau, \vec{\sigma}) - \boldsymbol{\epsilon}\, T_{\tau r}(\tau, \vec{\sigma}) \approx 0,
\end{aligned}$$

$$\rho_U(\tau, \vec{\sigma}) \;=\; \hat{U}^{\mu}(\tau)\, \rho_{\mu}(\tau, \vec{\sigma}), \qquad \rho_{Ur}(\tau, \vec{\sigma}) = \epsilon^{\mu}_r(\hat{U}(\tau))\, \rho_{\mu}(\tau, \vec{\sigma}). \tag{B11}$$

Here we introduced the notation $\mathcal{I} = \Big(A_A(\tau, \vec{\sigma}), \pi^A(\tau, \vec{\sigma})\Big)$ $[\{A_A(\tau, \vec{\sigma}), \pi^B(\tau, \vec{\sigma}')\} = \delta^{\beta}_{\alpha}\, \delta^3(\vec{\sigma} - \vec{\sigma}')]$ to denote the electro-magnetic canonical variables.

Introducing the variable (the *internal mass* of the electro-magnetic field on the simultaneity and Cauchy surface Σ_{τ})

$$M_U(\tau) = \int d^3\sigma\, \rho_U(\tau, \vec{\sigma}), \qquad \{\theta(\tau), M_U(\tau)\} = \boldsymbol{\epsilon}, \tag{B12}$$

the constraint (B9) can be written in the form

$$
\begin{aligned}
H_\perp(\tau) &= M_U(\tau) - \mathcal{E}[\mathcal{A}^r(\tau,\vec{\sigma}),\mathcal{I}] \approx 0,\\
\mathcal{E}[\mathcal{A}^r(\tau,\vec{\sigma}),\mathcal{I}] &= \int d^3\sigma\, [T_{\tau\tau}(z^\mu(\tau,\vec{\sigma}),\mathcal{I})]_{\text{evaluated on the gauge fixing}}.
\end{aligned}
\tag{B13}
$$

We also have

$$
\begin{aligned}
&\{\mathcal{A}^r(\tau,\vec{\sigma}),\rho_{Us}(\tau,\vec{\sigma}^{'})\} = -\boldsymbol{\epsilon}\,\delta^r_s\,\delta^3(\vec{\sigma}-\vec{\sigma}^{'}),\\
\rho^\mu(\tau,\vec{\sigma}) &\approx \boldsymbol{\epsilon}\Big[\rho_U(\tau,\vec{\sigma})\,\hat{U}^\mu(\tau) - \epsilon^\mu_r(\hat{U}(\tau))\,\rho_{Ur}(\tau,\vec{\sigma})\Big],\\
\rho_U(\tau,\vec{\sigma}) &\approx \boldsymbol{\epsilon}\,\tilde{T}_{\tau\tau}(\mathcal{A}^s(\tau,\vec{\sigma}),\mathcal{I}) = \boldsymbol{\epsilon}\,[T_{\tau\tau}(z^\mu(\tau,\vec{\sigma}),\mathcal{I})]_{\text{evaluated on the gauge fixing}}.
\end{aligned}
\tag{B14}
$$

Then we can rewrite the constraints $\mathcal{H}_r(\tau,\vec{\sigma}) \approx 0$ in the form

$$
\begin{aligned}
\mathcal{H}_r(\tau,\vec{\sigma}) &= \frac{\partial \mathcal{A}^s(\tau,\vec{\sigma})}{\partial\sigma^r}\,\rho_{Us}(\tau,\vec{\sigma}) - \boldsymbol{\epsilon}\,\tilde{T}_{\tau r}(\mathcal{A}^s(\tau,\vec{\sigma}),\mathcal{I}) \approx 0,\\
\tilde{T}_{\tau r}(\mathcal{A}^s(\tau,\vec{\sigma}),\mathcal{I}) &= [T_{\tau r}(z^\mu(\tau,\vec{\sigma}),\mathcal{I})]_{\text{evaluated on the gauge fixing}}.
\end{aligned}
\tag{B15}
$$

Eqs.(B13) show that the gauge fixing (B6) and the constraints $\mathcal{H}_U(\tau,\vec{\sigma}) - \delta^3(\vec{\sigma})\,\mathcal{H}_U(\tau) = \hat{U}^\mu(\tau)\,[\mathcal{H}_\mu(\tau,\vec{\sigma}) - \delta^3(\vec{\sigma})\int d^3\sigma_1\,\mathcal{H}_\mu(\tau,\vec{\sigma}_1)] = \hat{U}(\tau)\,\mathcal{H}_\mu(\tau,\vec{\sigma}) - H_\perp(\tau)\,\delta^3(\vec{\sigma}) \approx \tilde{\rho}_U(\tau,\vec{\sigma}) - \boldsymbol{\epsilon}\,[T_{\tau\tau}(\tau,\vec{\sigma}) - \delta^3(\vec{\sigma})\int d^3\sigma_1\,T_{\tau\tau}(\tau,\vec{\sigma}_1)] \approx 0$, with $\tilde{\rho}_U(\tau,\vec{\sigma}) = \rho_U(\tau,\vec{\sigma}) - \boldsymbol{\epsilon}\,M_U(\tau)$, form a pair of second class constraints and the surviving first class constraints are $H_\perp(\tau) \approx 0$ and $\mathcal{H}_r(\tau,\vec{\sigma}) \approx 0$.

After the gauge fixing (B6), a set of canonical variables for the reduced embedding are $\theta(\tau)$, $M_U(\tau)$, $\mathcal{A}^r(\tau,\vec{\sigma})$, $\rho_{Ur}(\tau,\vec{\sigma})$. Note that they have non zero Poisson brackets with $X^\mu(\tau)$, which therefore has to be replaced with a new $\tilde{X}^\mu(\tau)$ to complete the canonical basis with $\tilde{X}^\mu(\tau)$ and $U^\mu(\tau)$.

It can be shown [54] that the Dirac brackets associated to the gauge fixing (B6) are

[54] To show the validity of Eq.(B16), let us consider the constraints $\mathcal{H}_U(\tau) = \int d^3\sigma\,\mathcal{H}_U(\tau,\vec{\sigma}) =$

$$
\begin{aligned}
&\{A(\tau), B(\tau)\}^* \approx \{A(\tau), B(\tau)\} + \\
&+ \int d^3\sigma [\{A(\tau), S(\tau, \vec{\sigma})\}\{\mathcal{H}_U(\tau, \vec{\sigma}), B(\tau)\} - \{B(\tau), S(\tau, \vec{\sigma})\}\{\mathcal{H}_U(\tau, \vec{\sigma}), A(\tau)\}],
\end{aligned}
$$

$$
\Rightarrow \qquad \rho^\mu(\tau, \vec{\sigma}) \approx \tilde{T}_{\tau\tau}(\mathcal{A}^s(\tau, \vec{\sigma}), \mathcal{I})\, \hat{U}^\mu(\tau) - \boldsymbol{\epsilon}\, \epsilon^\mu_r(\hat{U}(\tau))\, \rho_{Ur}(\tau, \vec{\sigma}),
$$

$$
\begin{aligned}
P^\mu_s(\tau) \;&=\; \Big[\sqrt{\boldsymbol{\epsilon}\, U^2(\tau)} + \int d^3\sigma\, \tilde{T}_{\tau\tau}(\mathcal{A}^s(\tau, \vec{\sigma}), \mathcal{I})\Big]\, \hat{U}^\mu(\tau) - \boldsymbol{\epsilon}\, \epsilon^\mu_r(\hat{U}(\tau)) \int d^3\sigma\, \rho_{Ur}(\tau, \vec{\sigma}) \approx \\
&\approx\; [1 + M_U(\tau)]\, \hat{U}^\mu(\tau) - \boldsymbol{\epsilon}\, \epsilon^\mu_r(\hat{U}(\tau)) \int d^3\sigma\, \rho_{Ur}(\tau, \vec{\sigma}). \qquad \text{(B16)}
\end{aligned}
$$

It is easy to verify the following brackets [here $F(\mathcal{I})$ is a function of the canonical variables $\mathcal{I}$ only]

$$
\{F_1(\mathcal{I}), F_2(\mathcal{I})\}^* = \{F_1(\mathcal{I}), F_2(\mathcal{I})\},
$$

$$
\{\mathcal{A}^r(\tau, \vec{\sigma}), \mathcal{A}^s(\tau, \vec{\sigma}')\}^* = \{\rho_{Ur}(\tau, \vec{\sigma}), \rho_{Us}(\tau, \vec{\sigma}')\}^* = 0,
$$

$$
\{\mathcal{A}^r(\tau, \vec{\sigma}), \rho_{Us}(\tau, \vec{\sigma}')\}^* = -\boldsymbol{\epsilon}\, \delta^r_s\, \delta(\vec{\sigma} - \vec{\sigma}'),
$$

$$
\{\mathcal{A}^r(\tau, \vec{\sigma}), F(\mathcal{I})\}^* = \{\rho_{Ur}(\tau, \vec{\sigma}), F(\mathcal{I})\}^* = 0,
$$

$\hat{U}^\mu(\tau) \int d^3\sigma\, \mathcal{H}_\mu(\tau, \vec{\sigma}) \approx 0$ and $\mathcal{H}_{Ur}(\tau, \vec{\sigma}) = \epsilon^\mu_r(\hat{U}(\tau))\, \mathcal{H}_\mu(\tau, \vec{\sigma}) \approx 0$, which are weakly equal to $H_\perp(\tau) \approx 0$ and $\mathcal{H}_r(\tau, \vec{\sigma}) \approx 0$ when we add the gauge fixing $S(\tau, \vec{\sigma}) \approx 0$. These constraints have weakly vanishing Poisson brackets among themselves when $S(\tau, \vec{\sigma}) \approx 0$. We have $\{S(\tau, \vec{\sigma}), \mathcal{H}_U(\tau, \vec{\sigma}^{'}\} = \delta^3(\vec{\sigma}) - \delta^3(\vec{\sigma} - \vec{\sigma}^{'})$ [compatible with $S(\tau, \vec{0}) = 0$] and this implies $\{S(\tau, \vec{\sigma}), \mathcal{H}_U(0)\} = \int d^3\sigma^{'} \{S(\tau, \vec{\sigma}), \mathcal{H}_U(\tau, \vec{\sigma}^{'})\} = 0$ and $\{S(\tau, \vec{\sigma}), \mathcal{H}_U(\tau, \vec{\sigma}^{'} - \delta^3(\vec{\sigma}^{'})\, \mathcal{H}_U(\tau)\} = 0$. To find the Dirac brackets associated to the second class constraints $S(\tau, \vec{\sigma}) \approx 0$, $\mathcal{H}_U(\tau, \vec{\sigma}) - \delta^3(\vec{\sigma})\, \mathcal{H}_U(\tau) \approx 0$, we make their expansion around $\vec{\sigma} = 0$. Then the multipoles $\mathcal{H}_{m_1 m_2 m_3}(\tau) = \frac{1}{\sqrt{m_1!\, m_2!\, m_3!}} \int d^3\sigma\, (\sigma^1)^{m_1}\, (\sigma^2)^{m_2}\, (\sigma^3)^{m_3}\, \mathcal{H}_U(\tau, \vec{\sigma})$, $S_{m_1 m_2 m_3}(\tau) = \frac{1}{\sqrt{m_1!\, m_2!\, m_3!}} \frac{\partial^{m_1+m_2+m_3} S(\tau, \vec{\sigma})}{\partial\sigma^{1\, m_1} \partial\sigma^{2\, m_2} \partial\sigma^{3\, m_3}}|_{\vec{\sigma}=0}$, satisfy the algebra $\{S_{m_1 m_2 m_3}(\tau), \mathcal{H}_{n_1 n_2 n_3}(\tau)\} = \delta_{m_1 n_1}\, \delta_{m_2 n_2}\, \delta_{m_3 n_3}$, $\{S_{m_1 m_2 m_3}(\tau), S_{n_1 n_2 n_3}\} = 0$, $\{\mathcal{H}_{m_1 m_2 m_2}(\tau), \mathcal{H}_{n_1 n_2 n_3}(\tau)\} \approx 0$. This allows to get Eq.(B16) with $\mathcal{H}_U(\tau, \vec{\sigma})$ in place of $\mathcal{H}_\perp(\tau, \vec{\sigma})$. Then this result weakly implies Eq.(B16).

$$\{M_U(\tau), \theta(\tau)\}^* = \{M_U(\tau), \theta(\tau)\} = \boldsymbol{\epsilon},$$

$$\{M_U(\tau), M_U(\tau)\}^* = \{\theta(\tau), \theta(\tau)\} = 0,$$

$$\{M_U(\tau), F(\mathcal{I})\}^* = \{M_U(\tau), \mathcal{A}^r(\tau, \vec{\sigma})\}^* = \{M_U(\tau), \rho_{Ur}(\tau, \vec{\sigma})\}^* = 0,$$

$$\{\theta(\tau), F(\mathcal{I})\}^* = \{\theta(\tau), \mathcal{A}^r(\tau, \vec{\sigma})\}^* = \{\theta(\tau), \rho_{Ur}(\tau, \vec{\sigma})\}^* = 0. \qquad \text{(B17)}$$

Moreover we have

$$\{U^\mu(\tau), F(\mathcal{I})\}^* = \{U^\mu(\tau), \mathcal{A}^r(\tau, \vec{\sigma})\}^* = \{U^\mu(\tau), \rho_{Ur}(\tau, \vec{\sigma})\}^* =$$

$$= \{U^\mu(\tau), M_U(\tau)\}^* = \{U^\mu(\tau), \theta(\tau)\}^* = 0. \qquad \text{(B18)}$$

Since the Dirac brackets of $X^\mu(\tau)$ with the other canonical variables are very complicated, we do not give them.

All these brackets show us that the pairs $\theta(\tau)$, $M_U(\tau)$, $\mathcal{A}^r(\tau, \vec{\sigma})$, $\rho_{Ur}(\tau, \vec{\sigma})$, $\tilde{X}^\mu(\tau)$, $U^\mu(\tau)$ together with the original variables $\mathcal{I}$ are a canonical basis for the reduced phase space, if $\tilde{X}^\mu(\tau)$ is a suitable replacement of $X^\mu(\tau)$. To find $\tilde{X}^\mu(\tau)$ we have first to study the Lorentz covariance of the new variables on the reduced phase space.

Let us first observe that Eqs.(B6) imply

$$\{J^{\mu\nu}_s(\tau), S(\tau, \vec{\sigma})\} = \{J^{\mu\nu}_s(\tau), \mathcal{H}_\perp(\tau, \vec{\sigma})\} = 0. \qquad \text{(B19)}$$

so that the Dirac brackets (B16) change neither the Poisson algebra of the generators $J^{\mu\nu}$

$$\{J^{\mu\nu}_s(\tau), J^{\sigma\rho}_s(\tau)\}^* = \{J^{\mu\nu}_s(\tau), J^{\sigma\rho}_s(\tau)\} = C^{\mu\nu\sigma\rho}_{\alpha\beta} J^{\alpha\beta}_s(\tau),$$

$$C^{\mu\nu\rho\sigma}_{\alpha\beta} = \eta^\nu_\alpha \eta^\rho_\beta \eta^{\mu\sigma} + \eta^\mu_\alpha \eta^\sigma_\beta \eta^{\nu\rho} - \eta^\nu_\alpha \eta^\sigma_\beta \eta^{\mu\rho} - \eta^\mu_\alpha \eta^\rho_\beta \eta^{\nu\sigma}, \qquad \text{(B20)}$$

nor the transformations properties of the canonical variables on the reduced phase space. In particular we have the Lorentz scalar variables

$$\{J^{\mu\nu}_s(\tau), F(\mathcal{I})\}^* = \{J^{\mu\nu}_s(\tau), M_U(\tau)\}^* = \{J^{\mu\nu}_s(\tau), \theta(\tau)\}^* = 0. \tag{B21}$$

On the contrary the variables $\mathcal{A}^r(\tau,\vec{\sigma})$, $\rho^r_U(\tau,\vec{\sigma}) = \eta^{rs}\,\rho_{Us}(\tau,\vec{\sigma})$ are not scalar, but they transform as *Wigner spin-1 3-vectors* since the tetrad fields $\epsilon^\mu_A(\hat{U})$ are the columns of the standard Wigner boost $L(\hat{U},\hat{U}_o)$ for time-like Poincare' orbits.

In fact by using the infinitesimal transformations

$$\begin{aligned}
\Lambda_{\mu\nu} &= \eta_{\mu\nu} + \delta\omega_{\mu\nu}, \qquad \delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu},\\
R_{sr}(\Lambda,\hat{U}) &= \delta_{sr} + D_{sr}{}^{\mu\nu}(\hat{U})\,\delta\omega_{\mu\nu}, \qquad D_{sr}{}^{\mu\nu}(\hat{U}) = -D_{rs}{}^{\mu\nu}(\hat{U}) = -D_{sr}{}^{\nu\mu}(\hat{U}),
\end{aligned} \tag{B22}$$

in the last of Eqs.(B2), we obtain

$$(\eta_{\mu\nu} + \delta\omega_{\mu\nu})\;\epsilon^\nu_s(\hat{U})\;\Big(\delta_{sr} + D_{sr}{}^{\sigma\rho}(\hat{U})\,\delta\omega_{\sigma\rho}\Big) = \epsilon^\mu_r(\hat{U}) + \frac{1}{2}\delta\omega_{\sigma\rho}\{\epsilon^\mu_r(\hat{U}), J^{\sigma\rho}_s(\tau)\}. \tag{B23}$$

Then we get

$$\{\epsilon^\mu_r(\hat{U}), J^{\sigma\rho}_s(\tau)\} = -\eta^{\rho\mu}\epsilon^\sigma_r(\hat{U}) + \eta^{\sigma\mu}\epsilon^\rho_r(\hat{U}) + 2\,D_{sr}{}^{\sigma\rho}(\hat{U})\epsilon^\mu_s(\hat{U}), \tag{B24}$$

and finally

$$\begin{aligned}
\{\mathcal{A}^r(\tau,\vec{\sigma}), J^{\sigma\rho}_s(\tau)\}^* &= \{\epsilon^r_\mu(\hat{U})\,z^\mu(\tau,\vec{\sigma}), J^{\sigma\rho}_s(\tau)\} = -2\,D_{rs}{}^{\sigma\rho}(\hat{U})\,\mathcal{A}^s(\tau,\vec{\sigma}),\\
\{\rho^r_U(\tau,\vec{\sigma}), J^{\sigma\rho}_s(\tau)\}^* &= \{\epsilon^r_\mu(\hat{U})\,z^\mu(\tau,\vec{\sigma}), J^{\sigma\rho}_s(\tau)\} = -2\,D_{rs}{}^{\sigma\rho}(\hat{U})\,\rho^s_U(\tau,\vec{\sigma}).
\end{aligned} \tag{B25}$$

Since we have

$$\{\epsilon^\mu_r(\hat U), J^{\sigma\rho}_s(\tau)\} = \frac{\partial \epsilon^\mu_r(\hat U)}{\partial \hat U_\gamma}\, \{\hat U_\gamma(\tau), J^{\sigma\rho}_s(\tau)\} = -\frac{\partial \epsilon^\mu_r(\hat U)}{\partial \hat U_\gamma}\, \Big[\eta^\rho_\gamma\, \hat U^\sigma(\tau) - \eta^\sigma_\gamma\, \hat U^\rho(\tau)\Big], \qquad \text{(B26)}$$

we obtain the following expression for the matrix D

$$D^{\alpha\beta}_{rs}(\hat U) = \frac{1}{2}\left[\epsilon^\alpha_r(\hat U)\epsilon^\beta_s(\hat U) - \epsilon^\alpha_s(\hat U)\epsilon^\beta_r(\hat U) - \left(\hat U^\alpha \frac{\partial \epsilon^\mu_r(\hat U)}{\partial \hat U_\beta} - \hat U^\beta \frac{\partial \epsilon^\mu_r(\hat U)}{\partial \hat U_\alpha}\right)\, \epsilon_{s\mu}(\hat U)\right]. \qquad \text{(B27)}$$

To find the last canonical variables $\tilde X^\mu(\tau)$, let us define

$$L^{\mu\nu}(\tau) = J^{\mu\nu}_s(\tau) - D_{rs}{}^{\mu\nu}(\hat U) \int d^3\sigma\, \left[\mathcal{A}^r(\tau,\vec\sigma)\, \rho^s_U(\tau,\vec\sigma) - \mathcal{A}^s(\tau,\vec\sigma)\, \rho^r_U(\tau,\vec\sigma)\right]. \qquad \text{(B28)}$$

Then from Eq.(B5) we get

$$L^{\mu\nu}(\tau) = X^\mu(\tau)U^\nu(\tau) - X^\nu(\tau)U^\mu(\tau) + \widetilde I^{\mu\nu}(\tau), \qquad \text{(B29)}$$

with

$$\begin{aligned}
\widetilde I^{\mu\nu}(\tau) &= \int d^3\sigma\, \left[z^\mu(\tau,\vec\sigma)\rho^\nu(\tau,\vec\sigma) - z^\nu(\tau,\vec\sigma)\rho^\mu(\tau,\vec\sigma)\right]_{\text{evaluated on the gauge fixing}} - \\
&- D_{rs}{}^{\mu\nu}(\hat U) \int d^3\sigma\, \left[\mathcal{A}^r(\tau,\vec\sigma)\, \rho^s_U(\tau,\vec\sigma) - \mathcal{A}^s(\tau,\vec\sigma)\, \rho^r_U(\tau,\vec\sigma)\right] = \\
&= U^\mu(\tau) \left[\frac{1}{\sqrt{\boldsymbol{\epsilon}\, U^2(\tau)}} \int d^3\sigma\, \Big(\theta(\tau)\, \epsilon^\nu_r(\hat U(\tau))\rho^r_U(\tau,\vec\sigma) - \epsilon^\nu_r(\hat U(\tau))\, \mathcal{A}^r(\tau,\vec\sigma)\, \rho_U(\tau,\vec\sigma)\Big) + \right. \\
&+ \left. \frac{\partial \epsilon^\alpha_r(\hat U(\tau)}{\partial \hat U_\nu}\, \epsilon_{s\alpha}(\hat U(\tau)) \int d^3\sigma\, \mathcal{A}^r(\tau,\vec\sigma)\, \rho^s_U(\tau,\vec\sigma)\right] - (\mu \leftrightarrow \nu) = \\
&\overset{def}{=} U^\mu(\tau)\, \widetilde W^\nu(\tau) - (\mu \leftrightarrow \nu). \qquad \text{(B30)}
\end{aligned}$$

Therefore we get

$$\begin{aligned}
L^{\mu\nu}(\tau) &= (X^\mu(\tau) - \widetilde{W}^\mu(\tau))U^\nu(\tau) - (\mu \leftrightarrow \nu) = \tilde{X}^\mu(\tau)U^\nu(\tau) - \tilde{X}^\nu(\tau)U^\mu(\tau),\\
J^{\mu\nu}_s &= \tilde{X}^\mu(\tau)\, U^\nu(\tau) - \tilde{X}^\nu(\tau)\, U^\mu(\tau) + D_{rs}{}^{\mu\nu}(\hat{U}) \int d^3\sigma\, [\mathcal{A}^r\, \rho^s_U - \mathcal{A}^s\, \rho^r_U](\tau, \vec{\sigma}) =\\
&\stackrel{def}{=} \tilde{X}^\mu(\tau)\, U^\nu(\tau) - \tilde{X}^\nu(\tau)\, U^\mu(\tau) + \tilde{S}^{\mu\nu},
\end{aligned}$$

$$\begin{aligned}
&\{\tilde{S}^{\mu\nu}, \tilde{S}^{\alpha\beta}\} = C^{\mu\nu\alpha\beta}_{\rho\sigma}\, \tilde{S}^{\rho\sigma} + \Big(\frac{\partial D_{rs}{}^{\mu\nu}(\hat{U})}{\partial \hat{U}_\beta}\, U^\alpha - \frac{\partial D_{rs}{}^{\mu\nu}(\hat{U})}{\partial \hat{U}_\alpha}\, U^\beta -\\
&\quad - \frac{\partial D_{rs}{}^{\alpha\beta}(\hat{U})}{\partial \hat{U}_\nu}\, U^\mu + \frac{\partial D_{rs}{}^{\alpha\beta}(\hat{U})}{\partial \hat{U}_\mu}\, U^\nu\Big)\, S^{rs},\\
&S^{rs} = \int d^3\sigma\, (\mathcal{A}^r\, \rho^s_U - \mathcal{A}^s\, \rho^r_U)(\tau, \vec{\sigma}),
\end{aligned} \tag{B31}$$

where

$$\begin{aligned}
\tilde{X}^\mu(\tau) &= X^\mu(\tau) - \widetilde{W}^\mu(\tau) =\\
&= X^\mu(\tau) + \frac{1}{\sqrt{\epsilon\, U^2(\tau)}} \int d^3\sigma\, \Big(\theta(\tau)\, \epsilon^\mu_r(\hat{U}(\tau))\rho^r_U(\tau, \vec{\sigma}) - \epsilon^\mu_r(\hat{U}(\tau))\, \mathcal{A}^r(\tau, \vec{\sigma})\, \rho_U(\tau, \vec{\sigma})\Big) +\\
&+ \frac{\partial \epsilon^\alpha_r(\hat{U}(\tau))}{\partial \hat{U}_\mu}\, \epsilon_{s\alpha}(\hat{U}(\tau)) \int d^3\sigma\, \mathcal{A}^r(\tau, \vec{\sigma})\, \rho^s_U(\tau, \vec{\sigma})
\end{aligned} \tag{B32}$$

We see that $\tilde{S}^{\mu\nu}$ does not satisfy the right algebra for a spin tensor: this suggests that a further modification of $\tilde{X}^\mu(\tau)$ should be possible so to obtain a real spin tensor.

From Eqs.(B2) we get

$$U_\mu(\tau)\, \widetilde{W}^\mu(\tau) = 0\ \Rightarrow\ U_\mu(\tau)\, \tilde{X}^\mu(\tau) = U_\mu(\tau)\, X^\mu(\tau), \tag{B33}$$

so that we can write

$$\begin{aligned}
\tilde{X}^\mu(\tau) &= (\hat{U}^\sigma(\tau)\, X_\sigma(\tau))\, \hat{U}^\sigma(\tau) + L^{\mu\rho}(\tau)\hat{U}_\rho(\tau) \frac{1}{\sqrt{\epsilon\, U^2(\tau)}} =\\
&= (\hat{U}^\sigma(\tau)\, X_\sigma(\tau))\, \hat{U}^\sigma(\tau) + J^{\mu\rho}(\tau)\hat{U}_\rho(\tau) \frac{1}{\sqrt{\epsilon\, U^2(\tau)}} -\\
&- \frac{\partial \epsilon^\alpha_r(\hat{U}(\tau)}{\partial \hat{U}_\nu}\, \epsilon_{s\alpha}(\hat{U}) \int d^3\sigma\, [\mathcal{A}^r(\tau, \vec{\sigma})\, \rho^s_U(\tau, \vec{\sigma}) - \mathcal{A}^s(\tau, \vec{\sigma})\, \rho^r_U(\tau, \vec{\sigma})]\,.
\end{aligned} \tag{B34}$$

By construction we have

$$\{L^{\mu\nu}(\tau), F(\mathcal{I})\}^* = \{L^{\mu\nu}(\tau), \mathcal{A}^r(\tau,\vec{\sigma})\}^* = \{L^{\mu\nu}(\tau), \rho_{Ur}(\tau,\vec{\sigma})\}^* =$$

$$= \{L^{\mu\nu}(\tau), M_U(\tau)\}^* = \{L^{\mu\nu}(\tau), \theta(\tau)\}^* = 0,$$

$$\{L^{\mu\nu}(\tau), U^\sigma(\tau)\}^* = \eta^{\nu\sigma} U^\mu(\tau) - \eta^{\mu\sigma} U^\nu(\tau), \qquad \text{(B35)}$$

and then we can get

$$\{\tilde{X}^\mu(\tau), F(\mathcal{I})\}^* = \{\tilde{X}^\mu(\tau), \mathcal{A}^r(\tau,\vec{\sigma})\}^* = \{\tilde{X}^\mu(\tau), \rho_{Ur}(\tau,\vec{\sigma})\}^* =$$

$$= \{\tilde{X}^\mu(\tau), M_U(\tau)\}^* = \{\tilde{X}^\mu(\tau), \theta(\tau)\}^* = 0. \qquad \text{(B36)}$$

A long and tedious calculation allows to get

$$\begin{aligned}
\{\tilde{X}^\mu(\tau), \tilde{X}^\nu(\tau)\}^* &= 0,\\
\{\tilde{X}^\mu(\tau), U^\nu(\tau)\}^* &= -\eta^{\mu\nu},\\
\{L^{\mu\nu}(\tau), L^{\sigma\rho}(\tau)\}^* &= C^{\mu\nu\sigma\rho}_{\alpha\beta} L^{\alpha\beta}(\tau),\\
\{L^{\mu\nu}(\tau), \tilde{X}^\sigma(\tau)\}^* &= \eta^{\nu\sigma}\tilde{X}^\mu(\tau) - \eta^{\mu\sigma}\tilde{X}^\nu(\tau). \qquad \text{(B37)}
\end{aligned}$$

The looked for final pairs of canonical variables are given by $\tilde{X}^\mu(\tau)$, $U^\mu(\tau)$. Let us remark that $\tilde{X}^\mu$ is *not a Lorentz four-vector* since we have

$$\begin{aligned}
\{J^{\mu\nu}(\tau), \tilde{X}^\sigma(\tau)\}^* &= \eta^{\nu\sigma}\tilde{X}^\mu(\tau) - \eta^{\mu\sigma}\tilde{X}^\nu(\tau) +\\
&+ \frac{\partial D_{rs}{}^{\mu\nu}(\hat{U})}{\partial \hat{U}_\sigma} \int d^3\sigma\, \left[\mathcal{A}^r(\tau,\vec{\sigma})\,\rho^s_U(\tau,\vec{\sigma}) - \mathcal{A}^s(\tau,\vec{\sigma})\,\rho^r_U(\tau,\vec{\sigma})\right]. \qquad \text{(B38)}
\end{aligned}$$

Following Ref.[43] we can make the canonical transformation: $(\tilde{X}^\mu, U^\mu) \mapsto (\hat{U}_\mu \tilde{X}^\mu = \hat{U}_\mu X^\mu, \sqrt{\epsilon U^2} \approx 1)$, $(\vec{z}, \vec{k} = \vec{U}/\sqrt{\epsilon U^2} \approx \vec{U})$ with the Newton-Wigner-like non-covariant 3-vector $\vec{z} = \sqrt{\epsilon U^2}\, (\vec{\tilde{X}} - \tilde{X}_o\, \vec{U}/U^o) \approx \vec{\tilde{X}} - \tilde{X}^o\, \vec{U}/\sqrt{1 - \vec{U}^2}$.

In conclusion with these Dirac brackets the Poincare' algebra is still satisfied $[\{P^\mu_s, P^\nu_s\}^* = 0$, $\{P^\mu_s, J^{\rho\sigma}_s\}^* = \eta^{\mu\rho}\, P^\sigma_s - \eta^{\mu\sigma}\, P^\rho_s$, $\{J^{\mu\nu}_s, J^{\sigma\rho}_s\}^* = C^{\mu\nu\sigma\rho}_{\alpha\beta}\, J^{\alpha\beta}_s]$ and the final algebra of the surviving first class constraints is

$$
\begin{aligned}
&\{\mathcal{H}_r(\tau, \vec{\sigma}), \mathcal{H}_s(\tau, \vec{\sigma}^{\prime})\}^* = \mathcal{H}_r(\tau, \vec{\sigma}^{\prime}) \frac{\partial}{\partial \sigma^{\prime\, s}} \delta^3(\vec{\sigma} - \vec{\sigma}^{\prime}) - \mathcal{H}_s(\tau, \vec{\sigma}) \frac{\partial}{\partial \sigma^s} \delta^3(\vec{\sigma} - \vec{\sigma}^{\prime}),\\
&\{H_\perp(\tau), \mathcal{H}_r(\tau, \vec{\sigma})\}^* = 0.
\end{aligned}
\qquad \text{(B39)}
$$

If we want to recover the embedding (B1), i.e. Eq.(6.110), we must add the following gauge fixings to the first class constraints $H_\perp(\tau) \approx 0$ and $\mathcal{H}_r(\tau, \vec{\sigma}) \approx 0$

$$
\begin{aligned}
&\theta(\tau) - x_U(\tau) - \hat{U}_\mu(\tau)\, x^\mu(0) \approx 0,\\
&\mathcal{A}^r(\tau, \vec{\sigma}) - \xi^r_U(\tau, \vec{\sigma}) - \epsilon^r_\mu(\hat{U}(\tau))\, x^\mu(0) \approx 0,\\
&\Downarrow\\
&z^\mu(\tau, \vec{0}) = x^\mu_U(\tau),\\
&\rho_{Ur}(\tau, \vec{\sigma}) \approx \epsilon\, A^s_r(\tau, \vec{\sigma})\, \tilde{T}_{\tau s}(\xi^u_U(\tau, \vec{\sigma}) + \epsilon^u_\mu(\hat{U}(\tau))\, x^\mu(0), \mathcal{I}),\\
&\qquad \Big[A^s_r(\tau, \vec{\sigma})\, inverse\, of\, \frac{\partial \xi^r_U(\tau, \vec{\sigma})}{\partial \sigma^s} \Big],
\end{aligned}
$$

$$
\begin{aligned}
S^{rs} &= \int d^3\sigma \left[(\xi^r_U\, A^{sv} - \xi^s_U\, A^{rv})\, \tilde{T}_{\tau v}\right](\tau,\vec{\sigma}) + \\
&+ x^\mu(0) \left[\epsilon^r_\mu(\hat{U}(\tau)) \int d^3\sigma\, A^{sv}(\tau,\vec{\sigma}) - \epsilon^s_\mu(\hat{U}(\tau)) \int d^3\sigma\, A^{rv}(\tau,\vec{\sigma})\right] \\
&\tilde{T}_{\tau v}(\xi^u_U(\tau,\vec{\sigma}) + \epsilon^u_\mu(\hat{U}(\tau))\, x^\mu(0), \mathcal{I}),
\end{aligned}
$$

$$
\begin{aligned}
P^\mu_s(\tau) &\approx \Big[1 + \mathcal{E}[\xi^u_U(\tau,\vec{\sigma}) + \epsilon^u_\alpha(\hat{U}(\tau))\, x^\alpha(0), \mathcal{I}]\Big]\, \hat{U}^\mu(\tau) - \\
&- \epsilon^\mu_r(\hat{U}(\tau)) \int d^3\sigma\, A^s_r(\tau,\vec{\sigma})\, \tilde{T}_{\tau s}(\xi^u_U(\tau,\vec{\sigma}) + \epsilon^u_\alpha(\hat{U}(\tau))\, x^\alpha(0), \mathcal{I}). \qquad \text{(B40)}
\end{aligned}
$$

The stability of the gauge fixings (B40) and $\frac{d\hat{U}^\mu(\tau)}{d\tau} \,\circeq\, 0$ imply $\mu(\tau) = -\dot{\theta}(\tau) = -\dot{x}_U(\tau) = -\dot{x}^\mu_U(\tau)\, \hat{U}_\mu(\tau)$, $\lambda^r(\tau,\vec{\sigma}) = -\boldsymbol{\epsilon}\, \mathcal{A}^r_s(\tau,\vec{\sigma})\, \frac{\partial \mathcal{A}^s(\tau,\vec{\sigma})}{\partial\tau} = -\boldsymbol{\epsilon}\, \mathcal{A}^r_s(\tau,\vec{\sigma}) \left(\dot{x}^\mu_U(\tau)\, \epsilon_{s\mu}(\hat{U}(\tau)) + \frac{\partial \zeta^s(\tau,\vec{\sigma})}{\partial\tau}\right)$ for the Dirac multipliers appearing in the Dirac Hamiltonian (B9) and in the associated Hamilton equations. If we go to new Dirac brackets, in the new reduced phase space we get $H_D = \kappa(\tau)\, \chi(\tau) + (electro - magnetic\, constraints)$ and this Dirac Hamiltonian does not reproduce the just mentioned Hamilton equations after their restriction to Eqs.(B40) due to the explicit τ-dependence of the gauge fixings. As a consequence, in analogy to what was done to get Eqs.(6.81) and (6.84), we have to find the correct Hamiltonian ruling the evolution in the reduced phase space. A look at the Hamilton equations shows that this Hamiltonian is

$$
\begin{aligned}
H &= -\mu(\tau)\, \mathcal{E}[\xi^r_U(\tau,\vec{\sigma}) + \epsilon^r_\mu(\hat{U}(\tau))\, x^\mu(0), \mathcal{I}] - \\
&- \int d^3\sigma\, \lambda^r(\tau,\vec{\sigma})\, \tilde{T}_{\tau r}(\xi^u_U(\tau,\vec{\sigma}) + \epsilon^u_\mu(\hat{U}(\tau))\, x^\mu(0), \mathcal{I}) + \\
&+ \int d^3\sigma\, [\lambda_\tau(\tau,\vec{\sigma})\, \pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\, \Gamma(\tau,\vec{\sigma})] = \\
&\\
&= \dot{x}^\mu_U(\tau) \Big[\hat{U}_\mu(\tau)\, \mathcal{E}[\xi^r_U(\tau,\vec{\sigma}) + \epsilon^r_\mu(\hat{U}(\tau))\, x^\mu(0), \mathcal{I}] - \\
&- \epsilon_{r\mu}(\hat{U}(\tau)) \int d^3\sigma\, \tilde{T}_{\tau r}(\xi^u_U(\tau,\vec{\sigma}) + \epsilon^u_\mu(\hat{U}(\tau))\, x^\mu(0), \mathcal{I})\Big] + \\
&+ \int d^3\sigma\, \mathcal{A}^r_s(\tau,\vec{\sigma})\, \frac{\partial \zeta^s(\tau,\vec{\sigma})}{\partial\tau}\, \tilde{T}_{\tau r}(\xi^u_U(\tau,\vec{\sigma}) + \epsilon^u_\mu(\hat{U}(\tau))\, x^\mu(0), \mathcal{I}) + \\
&+ \int d^3\sigma\, [\lambda_\tau(\tau,\vec{\sigma})\, \pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\, \Gamma(\tau,\vec{\sigma})] =
\end{aligned}
$$

$$
\begin{aligned}
&= \dot{x}^\mu_U(\tau)\left[P_{s\mu} - \hat{U}_\mu(\tau)\right] + \\
&+ \int d^3\sigma\, \mathcal{A}^r_s(\tau,\vec{\sigma})\, \frac{\partial \zeta^s(\tau,\vec{\sigma})}{\partial\tau}\, \tilde{T}_{\tau r}(\xi^u_U(\tau,\vec{\sigma}) + \epsilon^u_\mu(\hat{U}(\tau))\, x^\mu(0), \mathcal{I}) + \\
&+ \int d^3\sigma\, [\lambda_\tau(\tau,\vec{\sigma})\, \pi^\tau(\tau,\vec{\sigma}) - A_\tau(\tau,\vec{\sigma})\, \Gamma(\tau,\vec{\sigma})].
\end{aligned}
\tag{B41}
$$

We find that, apart from the contribution of the remaining first class constraints, the *effective non-inertial Hamiltonian* ruling the τ-evolution seen by the (in general non-inertial) observer $x^\mu_U(\tau)$ (the centroid origin of the 3-coordinates) is the sum of the projection of the total 4-momentum along the 4-velocity of the observer (without the term pertaining to the decoupled unit mass particle it is the *effective internal mass*) plus a term induced by the differential rotation of the 3-coordinate system around the world-line of the observer (the potential of *inertial forces*).

To eliminate the constraint $\chi(\tau) = \boldsymbol{\epsilon}\, U^2(\tau) - 1 \approx 0$ we add the gauge fixing

$$
\begin{aligned}
&\hat{U}_\mu(\tau)\, \tilde{X}^\mu(\tau) - \boldsymbol{\epsilon}\, \theta(\tau) = \hat{U}_\mu(\tau)\, X^\mu(\tau) - \boldsymbol{\epsilon}\, \theta(\tau) \approx 0, \quad \Rightarrow \quad \kappa(\tau) = -\frac{\boldsymbol{\epsilon}}{2}\, \dot{\theta}(\tau), \\
&\Downarrow \\
&\tilde{X}^\mu(\tau) = z^\mu(\tau, \vec{\sigma}_{\tilde{X}}(\tau)), \quad for\ some \quad \vec{\sigma}_{\tilde{X}}(\tau), \\
&X^\mu(\tau) = z^\mu(\tau, \vec{\sigma}_X(\tau)), \quad for\ some \quad \vec{\sigma}_X(\tau),
\end{aligned}
$$

$$
\begin{aligned}
U^\mu(\tau) &= \left(\sqrt{1+\vec{k}^2}; k^i(\tau)\right) = \hat{U}^\mu(\vec{k}), \\
\tilde{X}^\mu(\tau) &= \Big(\sqrt{1+\vec{k}^2}\, [\boldsymbol{\epsilon}\, \theta(\tau) + \vec{k}(\tau)\cdot\vec{z}(\tau)]; \\
&\quad z^i(\tau) + k^i(\tau)\, [\boldsymbol{\epsilon}\, \theta(\tau) + \vec{k}(\tau)\cdot\vec{z}(\tau)]\Big) = z^\mu(\tau, \vec{\sigma}_{\tilde{X}}(\tau)), \\
&\\
&L^{ij} = z^i\, k^j - z^j\, k^i, \qquad L^{oi} = -L^{io} = -z^i\, \sqrt{1+\vec{k}^2}.
\end{aligned}
\tag{B42}
$$

After having introduced new Dirac brackets, the extra added point particle of unit mass is reduced to the decoupled non-evolving variables $\vec{z}$, $\vec{k}$ and the not yet determined $\vec{\sigma}_{\tilde{X}}(\tau)$

and $\vec{\sigma}_X(\tau)$ give the 3-location of $\tilde{X}^\mu(\tau)$ and $X^\mu(\tau)$, respectively, which do not coincide with the world-line $x^\mu_U(\tau)$ of the non-inertial observer. Now we get $\dot{\tilde{X}}^\mu(\tau) = \dot{\theta}(\tau)\,\hat{U}^\mu(\tau)$ and this determines $\vec{\sigma}_{\tilde{X}}(\tau)$ as solution of the equation $\frac{\partial \mathcal{A}^r(\tau, \vec{\sigma}_{\tilde{X}}(\tau))}{\partial \tau} + \frac{\partial \mathcal{A}^r(\tau,\vec{\sigma})}{\partial \sigma^s}|_{\vec{\sigma}=\vec{\sigma}_{\tilde{X}}(\tau)}\,\dot{\sigma}^s_{\tilde{X}}(\tau) = 0$.

Since Eq.(B35) remains true, we still have that under a Lorentz transformation Λ we get $U^\mu \mapsto \Lambda^\mu{}_\nu\,U^\nu$. Moreover, we still have $\dot{x}^\mu_U(\tau) = \dot{\theta}(\tau)\,\hat{U}^\mu(\tau) + \epsilon^\mu_r(\hat{U}(\tau))\,\dot{\mathcal{A}}^r(\tau,\vec{0})$, namely the 4-velocity of the non-inertial observer is *not orthogonal* to the hyper-planes Σ_τ.

Finally the embedding (4.1) with a fixed unit normal l^μ, implying the breaking of the action of Lorentz boosts, is obtained by adding by hand the first class constraints

$$\hat{U}^\mu(\vec{k}) - l^\mu \approx 0, \tag{B43}$$

which determine the non-evolving constant $\vec{k}$. The conjugate constant $\vec{z}$ can be eliminated with the non-covariant gauge fixings

$$\vec{z} \approx \vec{0}, \;\Rightarrow\; \tilde{X}^\mu(\tau) \approx \boldsymbol{\epsilon}\,\theta(\tau)\,\hat{U}^\mu(\tau). \tag{B44}$$

The constraints (B43) and (B44) eliminate the extra non-evolving degrees of freedom $\vec{k}$ and $\vec{z}$ of the added decoupled point particle, respectively.

At this stage only the electro-magnetic canonical variables are left and Eq.(B40) determine the Poincare' generators. It is not clear if in this case there is a non-inertial analogue of the internal Poincare' group of the rest-frame instant form.

To recover the rest-frame instant form, having the Wigner hyper-planes orthogonal to the total 4-momentum as simultaneity surfaces, we must require $\hat{U}^\mu(\tau) - p^\mu/\sqrt{\boldsymbol{\epsilon}\,p^2} \approx 0$ instead of Eq.(B43). Then from Eq.(B40) we get the rest-frame conditions $\epsilon^r_\mu(\hat{U})\,p^\mu \approx 0$ (whose gauge fixing is the vanishing (B44) of the internal center of mass $\vec{\sigma}_X(\tau) \approx 0$, see Ref.[43]) and the invariant mass $\mathcal{E} + 1$, which is the correct one if we neglect the constant extra mass 1.

The technology of this Appendix could be used to study a family of admissible embeddings, whose leaves are general space-like hyper-surfaces, closed under the action of the Lorentz group.

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